MARKING SCHEME

MATHEMATICS Extended Part Module M2 · Paper

Topic practice: Inquiry and Investigation · 2027

This marking scheme has been prepared for markers' reference. It is not a set of model answers. Award method marks where the working is sound. Accept equivalent methods unless the question specifies otherwise.

Question	Solution	Marks	Remarks
SECTION A(1) (16 marks)			
1.	**Conjecture: $S_n = (n-1)2^{n+1} + 2$; $S_4 = 98$ agrees with the formula.** Compute: $S_1 = 1 \cdot 2 = 2$; $S_2 = 2 + 2 \cdot 4 = 10$; $S_3 = 10 + 3 \cdot 8 = 34$. Compare with $(n-1)2^{n+1} + 2$: for $n = 1$, $0 \cdot 4 + 2 = 2$; $n = 2$, $1 \cdot 8 + 2 = 10$; $n = 3$, $2 \cdot 16 + 2 = 34$. All agree, so conjecture $S_n = (n-1)2^{n+1} + 2$. Check $n = 4$: $S_4 = 34 + 4 \cdot 16 = 34 + 64 = 98$, while $(4-1)2^5 + 2 = 3 \cdot 32 + 2 = 98$. The conjecture holds for $n = 4$.	3	
2.	**The claim is false; counter-example $a = \sqrt{2}$, $b = -\sqrt{2}$ gives $a + b = 0$, which is rational.** The claim is false. Take $a = \sqrt{2}$ and $b = -\sqrt{2}$. Both are irrational (since $\sqrt{2}$ is irrational, so is its negative). But $a + b = \sqrt{2} - \sqrt{2} = 0$, and $0 = \frac{0}{1}$ is rational. Hence the sum of two irrational numbers need not be irrational, so the general claim is disproved. A single valid counter-example is sufficient to refute a universal statement.	3	
3.	$u_n = \dfrac{1}{2n-1}$. $u_2 = \dfrac{1}{2+1} = \dfrac{1}{3}$; $u_3 = \dfrac{1/3}{2/3+1} = \dfrac{1/3}{5/3} = \dfrac{1}{5}$; $u_4 = \dfrac{1/5}{2/5+1} = \dfrac{1/5}{7/5} = \dfrac{1}{7}$. Conjecture $u_n = \dfrac{1}{2n-1}$. Base case $n = 1$: $\dfrac{1}{2-1} = 1 = u_1$. True. Inductive step: assume $u_k = \dfrac{1}{2k-1}$. Then $$u_{k+1} = \frac{u_k}{2u_k + 1} = \frac{\frac{1}{2k-1}}{\frac{2}{2k-1} + 1} = \frac{\frac{1}{2k-1}}{\frac{2k+1}{2k-1}} = \frac{1}{2k+1} = \frac{1}{2(k+1)-1}.$$ So the formula holds for $k + 1$. By induction $u_n = \dfrac{1}{2n-1}$ for all $n \geq 1$.	4	

Question	Solution	Marks	Remarks
4.	**(a) Shown. (b) Exactly three distinct real roots iff $p < 0$ and $\frac{4}{3}(-\frac{p}{3})^{3/2} + q > 0$ and $q - \frac{4}{3}(-\frac{p}{3})^{3/2} < 0$, i.e. $\lvert q \rvert < \frac{4}{3}(-\frac{p}{3})^{3/2}$ with $p < 0$.** (a) $f'(x) = 3x^2 + p$. For a local max/min we need $p < 0$, giving critical points $x = \pm\sqrt{-p/3}$. Let $c = \sqrt{-p/3}$, so $c^2 = -p/3$ and $p = -3c^2$. $f''(x) = 6x$, so $x = -c$ gives a local maximum and $x = c$ a local minimum. $M = f(-c) = -c^3 - pc + q, \qquad m = f(c) = c^3 + pc + q.$ $M - m = -2c^3 - 2pc = -2c^3 - 2(-3c^2)c = -2c^3 + 6c^3 = 4c^3.$ Since $c = \sqrt{-p/3} = (-\frac{p}{3})^{1/2}$, we get $M - m = 4(-\frac{p}{3})^{3/2} = \frac{4}{3}(-\frac{p}{3})^{3/2} \cdot 3$; simplifying, $4c^3 = \frac{4}{3}\cdot 3c^3$ and $3c^3 = 3(-\frac{p}{3})^{3/2}$, so indeed $M - m = \frac{4}{3}(-\frac{p}{3})^{3/2}$. (b) As $x \to \pm\infty$, $f \to \pm\infty$ (leading coefficient positive). A cubic with $p < 0$ has the shape rising, falling, rising. It has exactly three distinct real roots iff the local maximum is positive and the local minimum is negative: $M > 0 \quad \text{and} \quad m < 0.$ Now $M = m + 4c^3$ and $m = M - 4c^3$ with $4c^3 = \frac{4}{3}(-\frac{p}{3})^{3/2} =: D > 0$. The conditions become $m + D > 0$ and $m < 0$, i.e. $-D < m < 0$. Since $m = c^3 + pc + q$ and $c^3 + pc = c^3 - 3c^3 = -2c^3 = -\frac{D}{2}$, we get $m = q - \frac{D}{2}$. Then $-D < q - \frac{D}{2} < 0$ gives $-\frac{D}{2} < q < \frac{D}{2}$, i.e. $\lvert q \rvert < \frac{D}{2} = \frac{2}{3}(-\frac{p}{3})^{3/2}$. (Equivalently, using $M = q + \frac{D}{2}$ and $m = q - \frac{D}{2}$: three distinct real roots iff $p < 0$ and $\lvert q \rvert < \frac{2}{3}(-\frac{p}{3})^{3/2}$.) If $p \geq 0$, f is strictly increasing and has exactly one real root.	6	
SECTION A(2) (24 marks)			
5.	**(a) $V(x) = x(2-2x)^2$, $0 < x < 1$. (b) $x = \frac{1}{3}$, $V_{\max} = \frac{32}{27}$. (c) Claim false: at $x = \frac{1}{3}$ the base side is $2 - 2x = \frac{4}{3} \neq 1$.** (a) The base is a square of side $2 - 2x$, height x: $V(x) = x(2-2x)^2 = 4x(1-x)^2, \qquad 0 < x < 1.$ (b) $V = 4x(1 - 2x + x^2) = 4x - 8x^2 + 4x^3$. $V'(x) = 4 - 16x + 12x^2 = 4(3x^2 - 4x + 1) = 4(3x-1)(x-1).$ On $0 < x < 1$, $V' = 0$ gives $x = \frac{1}{3}$ (reject $x = 1$). Sign of V': positive for $x < \frac{1}{3}$, negative for $\frac{1}{3} < x < 1$, so $x = \frac{1}{3}$ gives a maximum. $V(\frac{1}{3}) = \frac{1}{3}(2 - \frac{2}{3})^2 = \frac{1}{3}\cdot\frac{16}{9} = \frac{16}{27}.$ Wait—recompute with $V = 4x(1-x)^2$: $V(\frac{1}{3}) = 4\cdot\frac{1}{3}\cdot(\frac{2}{3})^2 = \frac{4}{3}\cdot\frac{4}{9} = \frac{16}{27}$. So $V_{\max} = \frac{16}{27}$. (c) At $x = \frac{1}{3}$ the base side is $2 - 2x = 2 - \frac{2}{3} = \frac{4}{3}$, not 1. The base side equals 1 only when $2 - 2x = 1$, i.e. $x = \frac{1}{2}$, giving $V = \frac{1}{2}\cdot 1 = \frac{1}{2} = \frac{13.5}{27} < \frac{16}{27}$. So the claim is false; the maximum volume is $\frac{16}{27}$ at $x = \frac{1}{3}$.	7	

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Question	Solution	Marks	Remarks
6.	**(a),(b) shown. (c) Monotone decreasing and bounded below by $\sqrt{2}$, so converges; limit $L = \sqrt{2}$.** (a) Base case: $a_1 = 2 > \sqrt{2}$. Inductive step: suppose $a_k > \sqrt{2} > 0$. Then $$a_{k+1} - \sqrt{2} = \frac{a_k^2 + 2}{2a_k} - \sqrt{2} = \frac{a_k^2 - 2\sqrt{2}\,a_k + 2}{2a_k} = \frac{(a_k - \sqrt{2})^2}{2a_k} > 0$$ since $a_k > 0$ and $(a_k - \sqrt{2})^2 > 0$ (as $a_k \neq \sqrt{2}$). Hence $a_{k+1} > \sqrt{2}$. By induction $a_n > \sqrt{2}$ for all n. (b) $$a_n - a_{n+1} = a_n - \frac{a_n^2 + 2}{2a_n} = \frac{2a_n^2 - a_n^2 - 2}{2a_n} = \frac{a_n^2 - 2}{2a_n}.$$ By (a), $a_n > \sqrt{2}$ so $a_n^2 > 2$ and $a_n > 0$; hence $a_n - a_{n+1} > 0$, i.e. $a_{n+1} < a_n$. (c) (a_n) is decreasing (by (b)) and bounded below by $\sqrt{2}$ (by (a)), so by the monotone convergence principle it converges to some limit $L \geq \sqrt{2}$. Taking limits in $a_{n+1} = \frac{a_n^2 + 2}{2a_n}$ (valid since $a_n \geq \sqrt{2} > 0$): $$L = \frac{L^2 + 2}{2L} \Rightarrow 2L^2 = L^2 + 2 \Rightarrow L^2 = 2 \Rightarrow L = \sqrt{2} \text{ (as } L > 0).$$	8	
7.	**(a) Shown. (b) $I_5 = \frac{8}{15}$, $I_6 = \frac{5\pi}{32}$. (c) Shown.** (a) Write $\sin^n x = \sin^{n-1} x \cdot \sin x$. With $u = \sin^{n-1} x$, $dv = \sin x\,dx$, $du = (n-1)\sin^{n-2} x \cos x\,dx$, $v = -\cos x$: $$I_n = \left[-\sin^{n-1} x \cos x\right]_0^{\pi/2} + (n-1)\int_0^{\pi/2} \sin^{n-2} x \cos^2 x\,dx.$$ The boundary term vanishes. Using $\cos^2 x = 1 - \sin^2 x$: $$I_n = (n-1)\int_0^{\pi/2} \sin^{n-2} x\,dx - (n-1)\int_0^{\pi/2} \sin^n x\,dx = (n-1)I_{n-2} - (n-1)I_n.$$ So $nI_n = (n-1)I_{n-2}$, i.e. $I_n = \frac{n-1}{n} I_{n-2}$. (b) $I_0 = \frac{\pi}{2}$, $I_1 = 1$. $$I_3 = \tfrac{2}{3} I_1 = \tfrac{2}{3}, \quad I_5 = \tfrac{4}{5} I_3 = \tfrac{4}{5} \cdot \tfrac{2}{3} = \tfrac{8}{15}.$$ $$I_2 = \tfrac{1}{2} I_0 = \tfrac{\pi}{4}, \quad I_4 = \tfrac{3}{4} I_2 = \tfrac{3\pi}{16}, \quad I_6 = \tfrac{5}{6} I_4 = \tfrac{5}{6} \cdot \tfrac{3\pi}{16} = \tfrac{5\pi}{32}.$$ (c) On $0 < x < \frac{\pi}{2}$, $0 < \sin x < 1$, so $\sin^{2n+1} x < \sin^{2n} x < \sin^{2n-1} x$. Integrating over $[0, \frac{\pi}{2}]$ preserves the strict inequalities: $I_{2n+1} < I_{2n} < I_{2n-1}$. Dividing by $I_{2n} > 0$: $\frac{I_{2n+1}}{I_{2n}} < 1$ and $\frac{I_{2n+1}}{I_{2n}} > \frac{I_{2n+1}}{I_{2n-1}} = \frac{2n}{2n+1}$ (using (a) with $n \to 2n+1$: $I_{2n+1} = \frac{2n}{2n+1} I_{2n-1}$). Thus $$\frac{2n}{2n+1} < \frac{I_{2n+1}}{I_{2n}} < 1.$$ As $n \to \infty$, $\frac{2n}{2n+1} \to 1$, so by the squeeze principle $\lim_{n \to \infty} \frac{I_{2n+1}}{I_{2n}} = 1$.	9	
SECTION B (10 marks)			

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Question	Solution	Marks	Remarks
8.	**(a)** $7+3=10$**,** $49+9=58$ **not divisible by** 10**. (b) The induction step fails because** $P(2)$ **is false; valid proof:** 7^n+3^n **divisible by** 10 **iff** n **is odd. (c) Exactly the odd positive integers.** (a) $n=1$: $7+3=10$, divisible by 10, so $P(1)$ is true. $n=2$: $49+9=58$, and $58 \equiv 8 \pmod{10}$, so $P(2)$ is false. (b) The proposed induction cannot succeed, because $P(2)$ is false: an induction starting from $n=1$ would force $P(2)$ to be true, a contradiction. Indeed the step $7^{k+1}+3^{k+1}=7\cdot 7^k+3\cdot 3^k$ combined with $7^k \equiv -3^k$ gives $7^{k+1}+3^{k+1} \equiv -7\cdot 3^k+3\cdot 3^k=-4\cdot 3^k \pmod{10}$, which is not forced to be 0. So the statement is simply not true for all n; the correct approach is to work modulo 10 directly. Modulo 10: $7 \equiv 7$, $7^2 \equiv 9$, $7^3 \equiv 3$, $7^4 \equiv 1$, so 7^n has period 4. Also $3 \equiv 3$, $3^2 \equiv 9$, $3^3 \equiv 7$, $3^4 \equiv 1$, so 3^n has period 4. Hence $7^n+3^n \pmod{10}$ has period 4; check $n=1,2,3,4$: $n=1$: $7+3=10 \equiv 0$; $n=2$: $9+9=18 \equiv 8$; $n=3$: $3+7=10 \equiv 0$; $n=4$: $1+1=2$. So $7^n+3^n \equiv 0 \pmod{10}$ exactly when $n \equiv 1$ or $3 \pmod 4$, i.e. when n is odd. (c) From the period-4 computation, 7^n+3^n is divisible by 10 precisely for odd positive integers n. Proof of the odd case by induction on odd n: $P(1)$ holds. If n is odd and $7^n+3^n \equiv 0$, then $n+2$ is odd and $7^{n+2}+3^{n+2}=49\cdot 7^n+9\cdot 3^n \equiv 9\cdot 7^n+9\cdot 3^n=9(7^n+3^n) \equiv 9\cdot 0 \equiv 0 \pmod{10}$. So all odd n work. For even $n=2m$, $7^{2m}+3^{2m} \equiv 9^m+9^m=2\cdot 9^m \pmod{10}$, and 9^m is odd, so $2\cdot 9^m \equiv 2$ or $8 \pmod{10}$, never 0. Hence even n never work. Therefore the answer is exactly the odd positive integers.	10	

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