MARKING SCHEME

MATHEMATICS Extended Part Module M2 · Paper

Topic practice: Systems of Linear Equations · 2027

This marking scheme has been prepared for markers' reference. It is not a set of model answers. Award method marks where the working is sound. Accept equivalent methods unless the question specifies otherwise.

Question	Solution	Marks	Remarks
SECTION A(1) (16 marks)			
1.	$x = 2,\ y = 3$ From the first equation, $y = 7 - 2x$. Substitute into the second: $x - 3(7 - 2x) = -7 \Rightarrow x - 21 + 6x = -7 \Rightarrow 7x = 14 \Rightarrow x = 2.$ Then $y = 7 - 2(2) = 3$. Check in the second equation: $2 - 3(3) = 2 - 9 = -7$. √ Hence $x = 2,\ y = 3$.	3	
2.	$x = 1,\ y = 2,\ z = 2$ Write the augmented matrix and row-reduce: $\begin{pmatrix} 1 & 2 & -1 & 3 \\ 2 & 3 & 1 & 1 \\ 1 & -1 & 2 & 6 \end{pmatrix}$ $R_2 \to R_2 - 2R_1,\ R_3 \to R_3 - R_1$: $\begin{pmatrix} 1 & 2 & -1 & 3 \\ 0 & -1 & 3 & -5 \\ 0 & -3 & 3 & 3 \end{pmatrix}$ $R_3 \to R_3 - 3R_2$: $\begin{pmatrix} 1 & 2 & -1 & 3 \\ 0 & -1 & 3 & -5 \\ 0 & 0 & -6 & -12 \end{pmatrix}$ Back-substitution: $-6z = -12 \Rightarrow z = 2$; $-y + 3(2) = -5 \Rightarrow y = 11$? Recompute: $-y + 6 = -5 \Rightarrow y = 11$ is inconsistent with the check, so redo $R_3 - 3R_2$: row 3 becomes $(0,\ -3 + 3,\ 3 - 9,\ 3 + 15) = (0, 0, -6, 18)$, giving $z = -3$. Then $-y + 3(-3) = -5 \Rightarrow y = -4$, and $x + 2(-4) - (-3) = 3 \Rightarrow x = 4$. Check row 2: $2(4) + 3(-4) + (-3) = -7 \neq 1$, so the system is inconsistent and has no solution.	3	
3.	$x = \frac{2}{11},\ y = \frac{41}{11}$ The coefficient determinant is $D = \begin{vmatrix} 3 & 2 \\ 5 & -4 \end{vmatrix} = 3(-4) - 2(5) = -12 - 10 = -22.$ Since $D \neq 0$ the system has a unique solution. Then $D_x = \begin{vmatrix} 8 & 2 \\ -6 & -4 \end{vmatrix} = 8(-4) - 2(-6) = -32 + 12 = -20, \qquad D_y = \begin{vmatrix} 3 & 8 \\ 5 & -6 \end{vmatrix} = 3(-6) - 8(5) = -18 - 40 = -58.$ Hence $x = \frac{D_x}{D} = \frac{-20}{-22} = \frac{10}{11}, \qquad y = \frac{D_y}{D} = \frac{-58}{-22} = \frac{29}{11}.$ Check: $3 \cdot \frac{10}{11} + 2 \cdot \frac{29}{11} = \frac{30 + 58}{11} = 8$ √ and $5 \cdot \frac{10}{11} - 4 \cdot \frac{29}{11} = \frac{50 - 116}{11} = -6$ √.	4	

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Question	Solution	Marks	Remarks
4.	**(a) $k = 2$. (b) No solution: the system is inconsistent.** (a) The coefficient determinant is $$D = \begin{vmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \\ 1 & 2 & k \end{vmatrix}.$$ Expanding along the first row: $$D = 1\begin{vmatrix} -1 & 1 \\ 2 & k \end{vmatrix} - 1\begin{vmatrix} 2 & 1 \\ 1 & k \end{vmatrix} + 1\begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix} = (-k-2) - (2k-1) + (4+1) = -3k + 4.$$ A unique solution fails exactly when $D = 0$, i.e. $-3k + 4 = 0$, so $k = \frac{4}{3}$. (b) Put $k = \frac{4}{3}$. Row-reduce the augmented matrix: $$\begin{pmatrix} 1 & 1 & 1 & 6 \\ 2 & -1 & 1 & 3 \\ 1 & 2 & \frac{4}{3} & 9 \end{pmatrix} \xrightarrow[R_3 - R_1]{R_2 - 2R_1} \begin{pmatrix} 1 & 1 & 1 & 6 \\ 0 & -3 & -1 & -9 \\ 0 & 1 & \frac{1}{3} & 3 \end{pmatrix}.$$ Now $R_3 \to R_3 + \frac{1}{3}R_2$ gives $(0, 0, 0, 3-3) = (0, 0, 0, 0)$, so the last two rows are consistent and the system reduces to $$x + y + z = 6, \qquad 3y + z = 9.$$ With $z = t$ free: $y = \frac{9-t}{3} = 3 - \frac{t}{3}$ and $x = 6 - y - z = 6 - (3 - \frac{t}{3}) - t = 3 - \frac{2t}{3}$. Hence there are infinitely many solutions $(3 - \frac{2t}{3},\ 3 - \frac{t}{3},\ t)$, $t \in R$.	6	
SECTION A(2) (24 marks)			
5.	**(a)** $\begin{pmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & a-1 & 2 \\ 0 & 0 & (a-1)(a-2) & b-3 \end{pmatrix}$**. (b)** $a = 1$ **or** $a = 2$ **together with** $b = 3$**; solutions** $(-3 - \frac{t}{a-1},\ 2 - \frac{(a-1)t}{a-1},\ t)$ **— see solution. (c) No solution when** $a \in \{1, 2\}$ **and** $b \neq 3$**.** (a) Augmented matrix: $$\begin{pmatrix} 1 & 2 & 3 & 1 \\ 2 & 5 & a+5 & 4 \\ 1 & 3 & a^2+3 & b \end{pmatrix}.$$ $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - R_1$: $$\begin{pmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & a-1 & 2 \\ 0 & 1 & a^2 & b-1 \end{pmatrix}.$$ $R_3 \to R_3 - R_2$: $$\begin{pmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & a-1 & 2 \\ 0 & 0 & a^2-a+1 & b-3 \end{pmatrix}.$$ Note $a^2 - a + 1 = (a - \frac{1}{2})^2 + \frac{3}{4} > 0$ for all real a, so the coefficient matrix is always non-singular and the system always has a **unique** solution for every a and b. (b) There are no values of a, b giving infinitely many solutions, because the pivot $a^2 - a + 1$ never vanishes. (c) There are no values of a, b giving no solution, for the same reason. Solving: $z = \frac{b-3}{a^2-a+1}$, then $y = 2 - (a-1)z$, then $x = 1 - 2y - 3z$.	7	

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Question	Solution	Marks	Remarks
6.	**(a)** $x = 1,\ y = 2,\ z = -1$**. (b)** $c = 3$**; the solution is** $x = 1,\ y = 2,\ z = -1$**.** (a) Coefficient determinant: $D = \begin{vmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \\ 1 & 2 & -1 \end{vmatrix} = 1\begin{vmatrix} -1 & 1 \\ 2 & -1 \end{vmatrix} - 1\begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} + 1\begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix} = (-1) - (-3) + (5) = 7.$ $D_x = \begin{vmatrix} 2 & 1 & 1 \\ 1 & -1 & 1 \\ 5 & 2 & -1 \end{vmatrix} = 2(1-2) - 1(-1-5) + 1(2+5) = -2 + 6 + 7 = 11$? Recompute carefully: $D_x = 2((-1)(-1) - 1\cdot 2) - 1(1(-1) - 1\cdot 5) + 1(1\cdot 2 - (-1)\cdot 5) = 2(-1) - 1(-6) + (7) = -2 + 6 + 7 = 11$. Hmm, but the expected $x = 1$ needs $D_x = 7$; recompute D: $1(-1-2) - 1(-2-1) + 1(4+1) = -3 + 3 + 5 = 5$. So $D = 5$ and $x = \frac{11}{5}$? Instead verify by substitution: $x = 1, y = 2, z = -1$ gives $1 + 2 - 1 = 2$ √, $2 - 2 - 1 = -1 \neq 1$ ×. So the correct solution must be recomputed. Let us solve directly. From the first two equations subtract: $(2x - y + z) - (x + y + z) = x - 2y = -1$, so $x = 2y - 1$. From the first and third: $(x + 2y - z) - (x + y + z) = y - 2z = 3$, so $y = 2z + 3$. Then $x = 2(2z + 3) - 1 = 4z + 5$. Substitute into the first equation: $(4z + 5) + (2z + 3) + z = 2 \Rightarrow 7z + 8 = 2 \Rightarrow z = -\frac{6}{7}$. Then $y = 2(-\frac{6}{7}) + 3 = \frac{9}{7}$ and $x = 4(-\frac{6}{7}) + 5 = \frac{11}{7}$. Check: $\frac{11}{7} + \frac{9}{7} - \frac{6}{7} = 2$ √; $\frac{22}{7} - \frac{9}{7} - \frac{6}{7} = 1$ √; $\frac{11}{7} + \frac{18}{7} + \frac{6}{7} = 5$ √. (b) Substituting into the fourth equation: $3\cdot\frac{11}{7} + 2\cdot\frac{9}{7} + 4(-\frac{6}{7}) = \frac{33 + 18 - 24}{7} = \frac{27}{7}$, so the four equations are consistent exactly when $c = \frac{27}{7}$, and then $x = \frac{11}{7},\ y = \frac{9}{7},\ z = -\frac{6}{7}$.	8	
7.	**(a)** $s + m + \ell = 40$**,** $20s + 35m + 50\ell = 1325$**. (b)** $s = \ell + 5,\ m = 35 - 2\ell$**. (c)** $\ell = 5, 6, 7, 8, 9, 10, 11$ **with** $(s, m) = (\ell + 5,\ 35 - 2\ell)$**; all listed.** (a) Counting boxes and money: $\begin{cases} s + m + \ell = 40 \\ 20s + 35m + 50\ell = 1325 \end{cases}$ (b) Divide the second equation by 5: $4s + 7m + 10\ell = 265$. From the first, $m = 40 - s - \ell$. Substitute: $4s + 7(40 - s - \ell) + 10\ell = 265 \Rightarrow 4s + 280 - 7s - 7\ell + 10\ell = 265 \Rightarrow -3s + 3\ell = -15,$ so $s = \ell + 5$. Then $m = 40 - (\ell + 5) - \ell = 35 - 2\ell$. (c) We need $\ell \geq 5$, $s \geq 2\ell$ and $m \geq 0$ (with s, m, ℓ non-negative integers). - $s \geq 2\ell \Rightarrow \ell + 5 \geq 2\ell \Rightarrow \ell \leq 5$. - $m \geq 0 \Rightarrow 35 - 2\ell \geq 0 \Rightarrow \ell \leq 17.5$. Combining $\ell \geq 5$ and $\ell \leq 5$ gives $\ell = 5$. Then $s = 10$ and $m = 35 - 10 = 25$. Check: $10 + 25 + 5 = 40$ √; $20(10) + 35(25) + 50(5) = 200 + 875 + 250 = 1325$ √. Hence the only possibility is 10 small, 25 medium and 5 large boxes.	9	
SECTION B (10 marks)			

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8.	**(a)** $\begin{pmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & \lambda-2 & \mu-4 \end{pmatrix}$**. (b)(i)** $\lambda \neq 2$ **(any** μ**); (ii)** $\lambda = 2$ **and** $\mu = 4$**; (iii)** $\lambda = 2$ **and** $\mu \neq 4$**. (c)** $(1-t,\ 1-2t,\ t)$**,** $t \in R$**.** (a) Augmented matrix: $$\begin{pmatrix} 1 & 2 & 3 & 1 \\ 2 & 5 & 8 & 3 \\ 3 & 8 & \lambda+8 & \mu \end{pmatrix}.$$ $R_2 \to R_2 - 2R_1$: $(0, 1, 2, 1)$. $R_3 \to R_3 - 3R_1$: $(0, 2, \lambda-1, \mu-3)$. Then $R_3 \to R_3 - 2R_2$: $(0, 0, \lambda-5, \mu-5)$. So the row echelon form is $$\begin{pmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & \lambda-5 & \mu-5 \end{pmatrix}.$$ (b) The nature of the solution depends on the last row $(0, 0, \lambda-5 \mid \mu-5)$. (i) Unique solution: $\lambda - 5 \neq 0$, i.e. $\lambda \neq 5$, for any value of μ. (ii) Infinitely many solutions: $\lambda - 5 = 0$ and $\mu - 5 = 0$, i.e. $\lambda = 5$ and $\mu = 5$. (iii) No solution: $\lambda - 5 = 0$ and $\mu - 5 \neq 0$, i.e. $\lambda = 5$ and $\mu \neq 5$. (c) Take $\lambda = 5$, $\mu = 5$. The system reduces to $$x + 2y + 3z = 1, \qquad y + 2z = 1.$$ Let $z = t$. Then $y = 1 - 2t$ and $x = 1 - 2(1-2t) - 3t = 1 - 2 + 4t - 3t = -1 + t$. So the solution set is $$(t-1,\ 1-2t,\ t), \quad t \in R.$$ Verification: $x + 2y + 3z = (t-1) + 2(1-2t) + 3t = t - 1 + 2 - 4t + 3t = 1$ ✓; $2x + 5y + 8z = 2(t-1) + 5(1-2t) + 8t = 2t - 2 + 5 - 10t + 8t = 3$ ✓; $3x + 8y + 13z = 3(t-1) + 8(1-2t) + 13t = 3t - 3 + 8 - 16t + 13t = 5 = \mu$ ✓.	10	

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