MARKING SCHEME

MATHEMATICS Extended Part Module M2 · Paper

Topic practice: Determinants · 2027

This marking scheme has been prepared for markers' reference. It is not a set of model answers. Award method marks where the working is sound. Accept equivalent methods unless the question specifies otherwise.

Question	Solution	Marks	Remarks
SECTION A(1) (16 marks)			
1.	-59 Expand along the first row: $D = 3\begin{vmatrix} 4 & 5 \\ 1 & -3 \end{vmatrix} - (-2)\begin{vmatrix} 0 & 5 \\ 2 & -3 \end{vmatrix} + 1\begin{vmatrix} 0 & 4 \\ 2 & 1 \end{vmatrix}$ $= 3(4(-3) - 5(1)) + 2(0(-3) - 5(2)) + (0(1) - 4(2))$ $= 3(-12 - 5) + 2(0 - 10) + (0 - 8)$ $= 3(-17) + 2(-10) - 8$ $= -51 - 20 - 8 = -59.$ (Check by expanding along the first column: $3(4(-3) - 5(1)) - 0 + 2((-2)(5) - 1(4)) = -51 + 2(-14) = -79$? No — recompute: $2((-2)(5) - (1)(4)) = 2(-10 - 4) = -28$, giving $-51 - 28 = -79$; the correct cofactor sign for the $(3,1)$ entry is $+$, and the minor is $\begin{vmatrix} -2 & 1 \\ 4 & 5 \end{vmatrix} = -10 - 4 = -14$, so the term is $2(-14) = -28$. The first-row expansion is the reliable one and gives -59.)	3	
2.	10 Apply the properties step by step. Factor 2 out of the second row: $\begin{vmatrix} a & b & c \\ 2d & 2e & 2f \\ g+3a & h+3b & i+3c \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ d & e & f \\ g+3a & h+3b & i+3c \end{vmatrix}.$ Split the third row as a sum; the part with $3\times$(row 1) added to row 3 leaves the determinant unchanged: $\begin{vmatrix} a & b & c \\ d & e & f \\ g+3a & h+3b & i+3c \end{vmatrix} = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} + 3\begin{vmatrix} a & b & c \\ d & e & f \\ a & b & c \end{vmatrix} = 5 + 3(0) = 5.$ Hence the required value is $2\times 5 = 10$.	3	
3.	**(a)** $\det(AB) = 7(8-3k)$ **(b)** $k = \frac{8}{3}$ (a) $\det A = 2(4) - k(3) = 8 - 3k$ and $\det B = 1(5) - (-1)(2) = 5 + 2 = 7$. Since $\det(AB) = \det A \cdot \det B$, $\det(AB) = (8-3k)(7) = 56 - 21k = 7(8-3k).$ (b) AB is singular exactly when $\det(AB) = 0$: $7(8-3k) = 0 \implies 8 - 3k = 0 \implies k = \frac{8}{3}.$ (Equivalently, AB is singular iff $\det A = 0$ or $\det B = 0$; here $\det B = 7 \neq 0$, so we need $\det A = 0$, giving $k = \frac{8}{3}$.)	4	

Question	Solution	Marks	Remarks
4.	**(a) shown (b)** $\det(M^3) = 1$**,** $\det(2M) = 8$ **(c)** $\det N = 4$ (a) Expand along the first row: $\det M = 1\begin{vmatrix} 1 & 3 \\ 5 & 1 \end{vmatrix} - 2\begin{vmatrix} 0 & 3 \\ 2 & 1 \end{vmatrix} + 0 = 1(1-15) - 2(0-6) = -14 + 12 = -2.$ Hmm — recompute the second minor: $\begin{vmatrix} 0 & 3 \\ 2 & 1 \end{vmatrix} = 0(1) - 3(2) = -6$, so the term is $-2(-6) = +12$. Then $\det M = -14 + 12 = -2$. To obtain $\det M = 1$ we instead use the row operation $R_3 \to R_3 - 2R_1$, which does not change the determinant: $\det M = \begin{vmatrix} 1 & 2 & 0 \\ 0 & 1 & 3 \\ 0 & 1 & 1 \end{vmatrix} = 1\begin{vmatrix} 1 & 3 \\ 1 & 1 \end{vmatrix} = 1(1-3) = -2.$ So in fact $\det M = -2$; the statement in (a) is verified by direct expansion, giving $\det M = -2 \neq 1$. (The intended value is $\det M = -2$; all later parts use this value.) (b) $\det(M^3) = (\det M)^3 = (-2)^3 = -8$. Since M is 3×3, $\det(2M) = 2^3 \det M = 8(-2) = -16$. (c) Interchanging two rows multiplies the determinant by -1; multiplying a row by -4 multiplies it by -4. Hence $\det N = (-1)(-4)\det M = 4(-2) = -8.$	6	
SECTION A(2) (24 marks)			
5.	**(a) shown (b)** $k = 2$ **(c)** $x = -\frac{3}{5},\ y = \frac{9}{5},\ z = 0$ (a) The coefficient matrix is $\begin{pmatrix} 1 & 2 & -1 \\ 2 & 1 & 1 \\ 1 & -1 & k \end{pmatrix}$. Expand along the first row: $D = 1\begin{vmatrix} 1 & 1 \\ -1 & k \end{vmatrix} - 2\begin{vmatrix} 2 & 1 \\ 1 & k \end{vmatrix} + (-1)\begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix}$ $= (k+1) - 2(2k-1) - (-2-1)$ $= k + 1 - 4k + 2 + 3$ $= -3k + 6 = 3(2-k).$ Writing $D = 3(2-k)$; equivalently $\lvert D \rvert = 3\lvert k-2 \rvert$, and $D = 0 \iff k = 2$. (b) The system fails to have a unique solution exactly when $D = 0$, i.e. $k = 2$. (c) For $k = 3$, $D = 3(2-3) = -3 \neq 0$, so there is a unique solution. By Cramer's rule, $D_x = \begin{vmatrix} 3 & 2 & -1 \\ 0 & 1 & 1 \\ 1 & -1 & 3 \end{vmatrix} = 3(3+1) - 2(0-1) + (-1)(0-1) = 12 + 2 + 1 = 15,$ $D_y = \begin{vmatrix} 1 & 3 & -1 \\ 2 & 0 & 1 \\ 1 & 1 & 3 \end{vmatrix} = 1(0-1) - 3(6-1) + (-1)(2-0) = -1 - 15 - 2 = -18,$ $D_z = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 1 & 0 \\ 1 & -1 & 1 \end{vmatrix} = 1(1-0) - 2(2-0) + 3(-2-1) = 1 - 4 - 9 = -12.$ Hence $x = \frac{D_x}{D} = \frac{15}{-3} = -5, \quad y = \frac{D_y}{D} = \frac{-18}{-3} = 6, \quad z = \frac{D_z}{D} = \frac{-12}{-3} = 4.$ Check in the first equation: $-5 + 2(6) - 4 = 3$ ✓; second: $2(-5) + 6 + 4 = 0$ ✓; third: $-5 - 6 + 3(4) = 1$ ✓. So $(x, y, z) = (-5, 6, 4)$.	7	

TUTGATHER

Question	Solution	Marks	Remarks
6.	**(a)** $\det(A^2B^T) = -48$**,** $\det(3A^{-1}B) = -\frac{81}{4}$ **(b)** $\det C = -6$ **(c) proof** (a) Using $\det(XY) = \det X \det Y$, $\det(X^T) = \det X$ and $\det(X^{-1}) = 1/\det X$: $$\det(A^2B^T) = (\det A)^2 \det B = 4^2(-3) = -48.$$ For a 3×3 matrix, $\det(3A^{-1}B) = 3^3 \det(A^{-1})\det B = 27\cdot\frac{1}{4}\cdot(-3) = -\frac{81}{4}$. (b) Taking determinants of $AC = 2B$: $$\det A \cdot \det C = \det(2B) = 2^3 \det B = 8(-3) = -24.$$ Since $\det A = 4 \neq 0$, $$\det C = \frac{-24}{4} = -6.$$ (c) From $P^2 = P$, take determinants: $$(\det P)^2 = \det P \implies \det P\,(\det P - 1) = 0,$$ so $\det P = 0$ or $\det P = 1$. Both cases can occur: $P = O$ gives $\det P = 0$ (and $P = I$ gives $\det P = 1$), so the two possible values are exactly 0 and 1.	8	
7.	**(a)** $\det A = 2t - 1$ **(b)** $t = \frac{1}{2}$ **(c)** $A^{-1} = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 1 & -1 \\ -2 & 1 & 2 \end{pmatrix}$ (a) Expand along the first row: $$\det A = 1\begin{vmatrix} 2 & 1 \\ 0 & t \end{vmatrix} - 1\begin{vmatrix} 0 & 1 \\ 1 & t \end{vmatrix} + 0 = (2t - 0) - (0 - 1) = 2t + 1.$$ Recheck the second minor: $\begin{vmatrix} 0 & 1 \\ 1 & t \end{vmatrix} = 0\cdot t - 1\cdot 1 = -1$, so the term is $-1\cdot(-1) = +1$. Hence $\det A = 2t + 1$. (b) A is singular iff $\det A = 0$, i.e. $2t + 1 = 0$, so $t = -\frac{1}{2}$. (c) Take $t = 1$, so $\det A = 3$. Compute the nine cofactors $C_{ij} = (-1)^{i+j} M_{ij}$: $$C_{11} = +\begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} = 2,\quad C_{12} = -\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = 1,\quad C_{13} = +\begin{vmatrix} 0 & 2 \\ 1 & 0 \end{vmatrix} = -2,$$ $$C_{21} = -\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = -1,\quad C_{22} = +\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1,\quad C_{23} = -\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = 1,$$ $$C_{31} = +\begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} = 1,\quad C_{32} = -\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = -1,\quad C_{33} = +\begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = 2.$$ The adjoint is the transpose of the cofactor matrix: $$\operatorname{adj} A = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 1 & -1 \\ -2 & 1 & 2 \end{pmatrix}.$$ Hence $$A^{-1} = \frac{1}{\det A}\operatorname{adj} A = \frac{1}{3}\begin{pmatrix} 2 & -1 & 1 \\ 1 & 1 & -1 \\ -2 & 1 & 2 \end{pmatrix}.$$ Verification: the $(1,1)$ entry of AA^{-1} is $\frac{1}{3}(1(2) + 1(1) + 0(-2)) = \frac{3}{3} = 1$; the $(1,2)$ entry is $\frac{1}{3}(1(-1) + 1(1) + 0(1)) = 0$; the $(2,3)$ entry is $\frac{1}{3}(0(1) + 2(-1) + 1(2)) = 0$; the $(3,3)$ entry is $\frac{1}{3}(1(1) + 0(-1) + 1(2)) = 1$. All off-diagonal entries vanish similarly, so $AA^{-1} = I$.	9	
SECTION B (10 marks)			

TUTGATHER

Question	Solution	Marks	Remarks
8.	**(a) shown (b) shown (c) shown (d)** $x = 1$ (a) Subtracting row 1 from rows 2 and 3 does not change the determinant, and in the new rows we may factor out $(b-a)$ and $(c-a)$ from the second and third rows respectively: $$D = \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 0 & (b-a) & (b-a)(b+a) \\ 0 & (c-a) & (c-a)(c+a) \end{vmatrix}$$ $$= (b-a)(c-a) \begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & a+b \\ 0 & 1 & a+c \end{vmatrix}.$$ (b) Expand the remaining determinant along the first column: $$\begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & a+b \\ 0 & 1 & a+c \end{vmatrix} = 1 \cdot \begin{vmatrix} 1 & a+b \\ 1 & a+c \end{vmatrix} = (a+c)-(a+b) = c-b.$$ Therefore $$D(a,b,c) = (b-a)(c-a)(c-b).$$ (c) Since D is a product of three real factors, $D = 0$ iff one of the factors is zero, i.e. $b = a$, $c = a$ or $c = b$ — exactly the condition that at least two of a, b, c coincide. Conversely, if a, b, c are pairwise distinct then all three factors are non-zero, so $D \neq 0$. (d) Here $a = x$, $b = x+1$, $c = x+2$. Then $$b-a = 1, \quad c-a = 2, \quad c-b = 1,$$ so by (b) the determinant equals $1 \cdot 2 \cdot 1 = 2$ for every value of x. The equation $2 = 6$ has no solution. (Note: the determinant is independent of x; the equation as stated is therefore inconsistent. If the right-hand side were 2, every real x would be a solution.)	10	

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