MARKING SCHEME

MATHEMATICS Extended Part Module M2 · Paper

Topic practice: Applications of Definite Integrals · 2027

This marking scheme has been prepared for markers' reference. It is not a set of model answers. Award method marks where the working is sound. Accept equivalent methods unless the question specifies otherwise.

Question	Solution	Marks	Remarks
SECTION A(1) (16 marks)			
1.	$\frac{4}{3}$ The curve meets the x-axis where $x^2 - 4x + 3 = 0$, i.e. $(x-1)(x-3) = 0$, so $x = 1$ or $x = 3$. Between these roots the curve lies below the x-axis. Hence the area is $$A = -\int_1^3 (x^2 - 4x + 3)\,dx = -\left[\frac{x^3}{3} - 2x^2 + 3x\right]_1^3.$$ At $x = 3$: $9 - 18 + 9 = 0$. At $x = 1$: $\frac{1}{3} - 2 + 3 = \frac{4}{3}$. Thus $$A = -\left(0 - \frac{4}{3}\right) = \frac{4}{3}.$$	3	
2.	8π Using the disc method, the volume is $$V = \pi\int_0^4 (\sqrt{x})^2\,dx = \pi\int_0^4 x\,dx = \pi\left[\frac{x^2}{2}\right]_0^4 = \pi\cdot\frac{16}{2} = 8\pi.$$	3	
3.	8 **m** The particle changes direction when $v(t) = 0$: $3t(t-2) = 0$, so $t = 0$ or $t = 2$. For $0 < t < 2$, $v(t) < 0$; for $t > 2$, $v(t) > 0$. The displacement is $$s(t) = \int (3t^2 - 6t)\,dt = t^3 - 3t^2 + C.$$ Taking $s(0) = 0$, $s(t) = t^3 - 3t^2$. Then $s(2) = 8 - 12 = -4$ and $s(3) = 27 - 27 = 0$. The total distance is $$\lvert s(2) - s(0)\rvert + \lvert s(3) - s(2)\rvert = \lvert -4 - 0\rvert + \lvert 0 - (-4)\rvert = 4 + 4 = 8 \text{ m}.$$	4	
4.	$\frac{1}{3}$ Find the points of intersection: $x^2 = 2x - x^2 \Rightarrow 2x^2 - 2x = 0 \Rightarrow 2x(x-1) = 0$, so $x = 0$ or $x = 1$. On $[0, 1]$, $2x - x^2 \geq x^2$ because $2x - 2x^2 = 2x(1-x) \geq 0$. Thus the area is $$A = \int_0^1 [(2x - x^2) - x^2]\,dx = \int_0^1 (2x - 2x^2)\,dx = \left[x^2 - \frac{2x^3}{3}\right]_0^1 = 1 - \frac{2}{3} = \frac{1}{3}.$$	6	
SECTION A(2) (24 marks)			

TUTGATHER

Question	Solution	Marks	Remarks
5.	$\frac{128\pi}{15}$ Use the shell method. A vertical strip at distance $4-x$ from the axis of rotation, of height $\sqrt{x}$ and width dx, generates a cylindrical shell of volume $2\pi(4-x)\sqrt{x}\,dx$. Hence $$V = 2\pi\int_0^4 (4-x)\sqrt{x}\,dx = 2\pi\int_0^4 (4x^{1/2} - x^{3/2})\,dx.$$ Now $$\int_0^4 4x^{1/2}\,dx = 4\cdot\frac{2}{3}x^{3/2}\,\Big|_0^4 = \frac{8}{3}\cdot 8 = \frac{64}{3},$$ $$\int_0^4 x^{3/2}\,dx = \frac{2}{5}x^{5/2}\,\Big|_0^4 = \frac{2}{5}\cdot 32 = \frac{64}{5}.$$ Thus $$V = 2\pi(\frac{64}{3} - \frac{64}{5}) = 2\pi\cdot 64\cdot\frac{2}{15} = \frac{256\pi}{15}.$$ Wait, check: $\frac{64}{3} - \frac{64}{5} = 64\cdot\frac{2}{15} = \frac{128}{15}$. Then $V = 2\pi\cdot\frac{128}{15} = \frac{256\pi}{15}$. But the answer given is $\frac{128\pi}{15}$; there is a discrepancy. Let's recompute carefully. Actually, the shell method formula is $V = 2\pi\int_a^b$ (radius)(height) dx. Here radius $= 4-x$, height $= \sqrt{x}$. So $$V = 2\pi\int_0^4 (4-x)\sqrt{x}\,dx.$$ Compute the integral: $$\int_0^4 (4-x)\sqrt{x}\,dx = \int_0^4 (4x^{1/2} - x^{3/2})\,dx.$$ $$\int_0^4 4x^{1/2}\,dx = 4\cdot\frac{2}{3}[x^{3/2}]_0^4 = \frac{8}{3}\cdot 8 = \frac{64}{3}.$$ $$\int_0^4 x^{3/2}\,dx = \frac{2}{5}[x^{5/2}]_0^4 = \frac{2}{5}\cdot 32 = \frac{64}{5}.$$ So the integral equals $\frac{64}{3} - \frac{64}{5} = 64(\frac{1}{3} - \frac{1}{5}) = 64\cdot\frac{2}{15} = \frac{128}{15}$. Then $V = 2\pi\cdot\frac{128}{15} = \frac{256\pi}{15}$. But the answer stated is $\frac{128\pi}{15}$. That is half. Did I miss a factor? Let's check with the washer method. Rotating about $x = 4$, we can express x as a function of y: $x = y^2$ for $0 \le y \le 2$. The region is bounded by $x = 4$ on the right and $x = y^2$ on the left. When rotated about $x = 4$, the outer radius is $4 - y^2$ and inner radius is 0? Actually, the region is between $x = y^2$ and $x = 4$, for y from 0 to 2. When rotated about $x = 4$, the cross-section perpendicular to the axis of rotation (which is vertical) is a washer? Wait, the axis of rotation is vertical line $x = 4$. The region is to the left of $x = 4$. A horizontal strip at height y has width $4 - y^2$. When rotated about $x = 4$, it forms a disc of radius $4 - y^2$? No, the strip is horizontal, so rotating it about a vertical axis generates a cylindrical shell? Actually, if we take a horizontal strip and rotate about a vertical axis, the strip is parallel to the axis? No, the strip is horizontal, the axis is vertical, so they are perpendicular. Rotating a horizontal strip about a vertical axis generates a washer? Let's think: The region is bounded by $x = y^2$ (left), $x = 4$ (right), and $y = 0$ (bottom). For a given y, the cross-section perpendicular to the y-axis is a horizontal segment from $x = y^2$ to $x = 4$. When rotated about $x = 4$, this segment sweeps out an annulus (washer) with outer radius $4 - y^2$ and inner radius 0? Actually, the distance from a point (x, y) to the line $x = 4$ is $4 - x$. So the segment from $x = y^2$ to $x = 4$ has distances from 4 ranging from 0 (at $x = 4$) to $4 - y^2$ (at $x = y^2$). So rotating gives a disc of radius $4 - y^2$. So the volume is $$V = \pi\int_0^2 (4 - y^2)^2\,dy.$$ Compute: $$(4 - y^2)^2 = 16 - 8y^2 + y^4.$$ $$\int_0^2 (16 - 8y^2 + y^4)\,dy = [16y - \frac{8y^3}{3} + \frac{y^5}{5}]_0^2 = 32 - \frac{64}{3} + \frac{32}{5}.$$ Common denominator 15: $32 = \frac{480}{15}$, $\frac{64}{3} = \frac{320}{15}$, $\frac{32}{5} = \frac{96}{15}$. So $\frac{480 - 320 + 96}{15} = \frac{256}{15}$. Then $V = \pi\cdot\frac{256}{15} = \frac{256\pi}{15}$. So the correct volume is $\frac{256\pi}{15}$, not $\frac{128\pi}{15}$. I must have made a mistake in the answer. Let's re-read the question: "The region R is bounded by the curve $y = \sqrt{x}$, the line $x = 4$ and the x-axis. R is revolved about the line $x = 4$ to form a solid. Find the volume of the solid." That is exactly what I computed. So the answer should be $\frac{256\pi}{15}$. But the provided answer is $\frac{128\pi}{15}$. Perhaps the intended region was bounded by $y = \sqrt{x}$, $x = 4$ and the y-axis? No, that would be different. Or maybe the axis of rotation is $y = 0$? That would give 8π. So I think the answer is $\frac{256\pi}{15}$. I'll correct the answer and solution accordingly. Thus the answer is $\frac{256\pi}{15}$.	7	

TUTGATHER

Question	Solution	Marks	Remarks
6.	$\approx 1.48 \times 10^6$ **J** The tank is formed by revolving $y = \sqrt{x}$ about the x-axis, so its cross-section at distance x from the vertex is a circle of radius $\sqrt{x}$. The depth of water is 3 m, so the water occupies $0 \le x \le 3$. Consider a thin slice of water at position x with thickness dx. Its volume is $dV = \pi(\sqrt{x})^2\,dx = \pi x\,dx$, so its mass is $dm = 1000\pi x\,dx$ kg. This slice must be lifted to the top of the tank, which is at $x = 4$. The distance lifted is $4 - x$ m. The work done is $$W = \int_0^3 (1000\pi x)(9.8)(4 - x)\,dx = 9800\pi \int_0^3 (4x - x^2)\,dx.$$ Compute the integral: $$\int_0^3 (4x - x^2)\,dx = [2x^2 - \frac{x^3}{3}]_0^3 = 18 - 9 = 9.$$ Thus $W = 9800\pi \cdot 9 = 88200\pi$ J. Numerically, $88200\pi \approx 88200 \times 3.14159 \approx 2.77 \times 10^5$ J? Wait, $88200 \times 3.14159 \approx 277,000$ J, which is 2.77×10^5 J. But the answer given is 1.48×10^6 J. There is a discrepancy. Let's recompute: $9800 \times 9 = 88200$, times π is about 277,000. That is 2.77×10^5 J. But the answer says 1.48×10^6 J. That is about 5.3 times larger. Perhaps I used the wrong density? Density of water is 1000 kg/m^3, $g = 9.8$, so weight density is 9800 N/m^3. That seems correct. Maybe the tank is filled to depth 3 m, but the top is at $x = 4$, so distance lifted is $4 - x$. That is correct. Let's check the volume: $V = \pi \int_0^3 x\,dx = \pi \cdot 9/2 = 4.5\pi \approx 14.14$ m^3. Mass is about 14137 kg. Weight is about 138,500 N. Average lift distance is about 2.5 m, so work is about 346,000 J. So 2.77×10^5 J is plausible. The given answer 1.48×10^6 J is too high. Perhaps the tank is filled to depth 4 m? Then work would be $9800\pi \int_0^4 (4x - x^2)\,dx = 9800\pi[2x^2 - x^3/3]_0^4 = 9800\pi(32 - 64/3) = 9800\pi(32/3) \approx 9800 \times 3.14159 \times 10.6667 \approx 328,000$ J, still not 1.48×10^6. Maybe the density is 1000 kg/m^3 and $g = 9.8$, but the answer might have used $g = 9.8$ and density 1000, but perhaps the tank shape is different? Or maybe the curve is $y = x^2$? Let's check: if $y = x^2$, then radius is x^2, volume element $\pi x^4\,dx$, mass $1000\pi x^4\,dx$, lift distance $4 - x$, work $9800\pi \int_0^3 (4x^4 - x^5)\,dx = 9800\pi[4x^5/5 - x^6/6]_0^3 = 9800\pi(4 \cdot 243/5 - 729/6) = 9800\pi(194.4 - 121.5) = 9800\pi(72.9) \approx 2.24 \times 10^6$ J. That is closer but not 1.48×10^6. If the depth is 2 m, then work $9800\pi \int_0^2 (4x - x^2)\,dx = 9800\pi[2x^2 - x^3/3]_0^2 = 9800\pi(8 - 8/3) = 9800\pi(16/3) \approx 164,000$ J. So I think the answer 1.48×10^6 J is incorrect. I'll correct it to 2.77×10^5 J. Thus the answer is $\approx 2.77 \times 10^5$ J.	8	
7.	**(a)** 1 **(b)** $\frac{\pi}{2}(e^2 - 1)$ (a) The curve $y = \ln x$ meets the x-axis at $x = 1$. Thus $$A = \int_1^e \ln x\,dx.$$ Using integration by parts with $u = \ln x$, $dv = dx$, we get $\int \ln x\,dx = x\ln x - x + C$. Hence $$A = [x\ln x - x]_1^e = (e\ln e - e) - (1\ln 1 - 1) = (e - e) - (0 - 1) = 1.$$ (b) Using the shell method, a vertical strip at distance x from the y-axis, of height $\ln x$ and width dx, generates a shell of volume $2\pi x \ln x\,dx$. Thus $$V = 2\pi \int_1^e x\ln x\,dx.$$ Integrate by parts with $u = \ln x$, $dv = x\,dx$, so $du = dx/x$, $v = x^2/2$: $$\int x\ln x\,dx = \frac{x^2}{2}\ln x - \int \frac{x}{2}\,dx = \frac{x^2}{2}\ln x - \frac{x^2}{4} + C.$$ Therefore $$V = 2\pi[\frac{x^2}{2}\ln x - \frac{x^2}{4}]_1^e = 2\pi((\frac{e^2}{2} - \frac{e^2}{4}) - (0 - \frac{1}{4})) = 2\pi(\frac{e^2}{4} + \frac{1}{4}) = \frac{\pi}{2}(e^2 + 1).$$ Wait, check: at $x = e$, $\frac{e^2}{2}\ln e - \frac{e^2}{4} = \frac{e^2}{2} - \frac{e^2}{4} = \frac{e^2}{4}$. At $x = 1$, $\frac{1}{2}\ln 1 - \frac{1}{4} = -\frac{1}{4}$. So difference is $\frac{e^2}{4} - (-\frac{1}{4}) = \frac{e^2+1}{4}$. Then $V = 2\pi \cdot \frac{e^2+1}{4} = \frac{\pi}{2}(e^2 + 1)$. But the answer given is $\frac{\pi}{2}(e^2 - 1)$. There is a sign error. Let's recompute the integral: $\int x\ln x\,dx = \frac{x^2}{2}\ln x - \frac{x^2}{4}$. At $x = e$: $\frac{e^2}{2} - \frac{e^2}{4} = \frac{e^2}{4}$. At $x = 1$: $0 - \frac{1}{4} = -\frac{1}{4}$. So the definite integral is $\frac{e^2}{4} - (-\frac{1}{4}) = \frac{e^2+1}{4}$. So $V = 2\pi \cdot \frac{e^2+1}{4} = \frac{\pi}{2}(e^2 + 1)$. So the correct answer is $\frac{\pi}{2}(e^2 + 1)$, not $\frac{\pi}{2}(e^2 - 1)$. I'll correct it. Thus the answer is (a) 1 (b) $\frac{\pi}{2}(e^2 + 1)$.	9	
SECTION B (10 marks)			

TUTGATHER

Question	Solution	Marks	Remarks
8.	$\frac{\pi}{2}(\pi + 8)$ Use the washer method. The axis of rotation is $y = -1$. A vertical strip at position x has top $y = \sin x$ and bottom $y = 0$. When rotated about $y = -1$, the outer radius is $\sin x - (-1) = \sin x + 1$ and the inner radius is $0 - (-1) = 1$. Thus the cross-sectional area is $$A(x) = \pi[(\sin x + 1)^2 - 1^2] = \pi(\sin^2 x + 2\sin x).$$ Hence the volume is $$V = \pi \int_0^{\pi} (\sin^2 x + 2\sin x)\, dx.$$ Now $$\int_0^{\pi} \sin^2 x\, dx = \int_0^{\pi} \frac{1 - \cos 2x}{2}\, dx = \left[\frac{x}{2} - \frac{\sin 2x}{4}\right]_0^{\pi} = \frac{\pi}{2},$$ and $$\int_0^{\pi} 2\sin x\, dx = [-2\cos x]_0^{\pi} = -2(-1) - (-2)(1) = 2 + 2 = 4.$$ Therefore $$V = \pi\left(\frac{\pi}{2} + 4\right) = \frac{\pi}{2}(\pi + 8).$$	10	

TUTGATHER