MARKING SCHEME

MATHEMATICS Extended Part Module M2 · Paper

Topic practice: Limits · 2027

This marking scheme has been prepared for markers' reference. It is not a set of model answers. Award method marks where the working is sound. Accept equivalent methods unless the question specifies otherwise.

Question	Solution	Marks	Remarks
SECTION A(1) (16 marks)			
1.	6 Direct substitution gives $\frac{0}{0}$, an indeterminate form, so we must simplify first. Factorise the numerator: $x^2 - 9 = (x-3)(x+3)$. For $x \neq 3$, $$\frac{x^2-9}{x-3} = \frac{(x-3)(x+3)}{x-3} = x+3.$$ Hence $$\lim_{x\to 3}\frac{x^2-9}{x-3} = \lim_{x\to 3}(x+3) = 3+3 = 6.$$ The limit exists even though the function is undefined at $x = 3$; the limit describes behaviour near $x = 3$, not at $x = 3$.	3	
2.	**(a)** 13 **(b)** $-\frac{15}{2}$ (a) By the scalar-multiple and difference rules for limits, $$\lim_{x\to 2}[2f(x)-g(x)] = 2\lim_{x\to 2}f(x) - \lim_{x\to 2}g(x) = 2(5)-(-3) = 13.$$ (b) By the product and sum rules, the numerator has limit $5\times(-3) = -15$ and the denominator has limit $5+(-3) = 2$. Since the limit of the denominator is non-zero, the quotient rule applies: $$\lim_{x\to 2}\frac{f(x)g(x)}{f(x)+g(x)} = \frac{-15}{2} = -\frac{15}{2}.$$	3	
3.	$k = 3$ For continuity at $x = 1$ we need $\lim_{x\to 1}h(x) = h(1) = k$. Left-hand limit: for $x < 1$, $$\frac{x^2-1}{x-1} = \frac{(x-1)(x+1)}{x-1} = x+1,$$ so $\lim_{x\to 1^-}h(x) = 1+1 = 2$. Right-hand limit: $\lim_{x\to 1^+}h(x) = \lim_{x\to 1^+}(2x+1) = 3$. Since $2 \neq 3$, the two one-sided limits differ, so $\lim_{x\to 1}h(x)$ does not exist for any choice of k. Hence there is no value of k making h continuous at $x = 1$. (Note: the jump in the graph cannot be removed by redefining a single point.)	4	

TUTGATHER

Question	Solution	Marks	Remarks
4.	**(a) $\frac{3}{2}$ (b) horizontal asymptote $y = \frac{3}{2}$** (a) Divide numerator and denominator by x^2: $$\frac{3x^2 - 5x + 2}{2x^2 + 7} = \frac{3 - \frac{5}{x} + \frac{2}{x^2}}{2 + \frac{7}{x^2}}.$$ As $x \to \infty$, each of $\frac{5}{x}, \frac{2}{x^2}, \frac{7}{x^2}$ tends to 0. By the limit theorems, $$\lim_{x \to \infty} \frac{3x^2 - 5x + 2}{2x^2 + 7} = \frac{3 - 0 + 0}{2 + 0} = \frac{3}{2}.$$ (b) As $x \to \infty$ (and likewise as $x \to -\infty$) the value of the function approaches $\frac{3}{2}$, so the graph flattens towards the line $y = \frac{3}{2}$. The horizontal asymptote is $y = \frac{3}{2}$.	6	
SECTION A(2) (24 marks)			
5.	**(b) $\frac{1}{4}$ (c) $\frac{3}{4}$** (a) Rationalise the numerator by multiplying by the conjugate $\sqrt{4+h} + 2$: $$\sqrt{4+h} - 2 = \frac{(\sqrt{4+h} - 2)(\sqrt{4+h} + 2)}{\sqrt{4+h} + 2} = \frac{(4+h) - 4}{\sqrt{4+h} + 2} = \frac{h}{\sqrt{4+h} + 2}.$$ This is valid for $h \neq 0$ (and $h > -4$). (b) For $h \neq 0$, $$\frac{\sqrt{4+h} - 2}{h} = \frac{1}{\sqrt{4+h} + 2}.$$ As $h \to 0$, $\sqrt{4+h} \to 2$, so by the quotient rule for limits $$\lim_{h \to 0} \frac{\sqrt{4+h} - 2}{h} = \frac{1}{2 + 2} = \frac{1}{4}.$$ (c) Put $h = 3x$. As $x \to 0$, $h \to 0$, and $$\frac{\sqrt{4+3x} - 2}{x} = 3 \cdot \frac{\sqrt{4+h} - 2}{h}.$$ Hence $$\lim_{x \to 0} \frac{\sqrt{4+3x} - 2}{x} = 3 \cdot \frac{1}{4} = \frac{3}{4}.$$	7	

TUTGATHER

Question	Solution	Marks	Remarks
6.	**(a)** $a + 2$ **(b)** $a = 0,\ b = 2$ **(c) not differentiable** (a) For the left-hand limit to exist as a finite number, the numerator must vanish at $x = 1$ (otherwise the quotient blows up). So $1 + a - 3 = 0$, giving $a = 2$; but we keep a general and require $x^2 + ax - 3$ to be divisible by $x - 1$. With $a = 2$: $x^2 + 2x - 3 = (x - 1)(x + 3)$, so for $x < 1$, $$f(x) = \frac{(x-1)(x+3)}{x-1} = x + 3,$$ and $\lim_{x \to 1^-} f(x) = 1 + 3 = 4$. More generally, writing $x^2 + ax - 3 = (x - 1)(x + c)$ gives $c - a = 1$ and $-c = -3$, so $c = 3$ and $a = 2$. Then the left-hand limit is $1 + c = 4 = a + 2$. (b) Right-hand limit: for $x > 1$, $\frac{x^2 - 1}{x - 1} = x + 1$, so $\lim_{x \to 1^+} f(x) = 2$. For continuity we need the two one-sided limits equal and equal to $f(1) = b$. But the left-hand limit is 4 and the right-hand limit is 2; these can never be equal. Hence no values of a and b make f continuous at $x = 1$. (c) Since f is not continuous at $x = 1$, it cannot be differentiable there (differentiability implies continuity). So f is not differentiable at $x = 1$.	8	
7.	**(a)** 5 **and** −5 **(b) no (c)** 2 **and** −2 First factorise: $2x^2 + x - 3 = (2x + 3)(x - 1)$. (a) For $x > 1$, $\lvert x - 1 \rvert = x - 1$, so $$f(x) = \frac{(2x+3)(x-1)}{x-1} = 2x + 3,$$ and $\lim_{x \to 1^+} f(x) = 2(1) + 3 = 5$. For $x < 1$, $\lvert x - 1 \rvert = -(x - 1)$, so $$f(x) = \frac{(2x+3)(x-1)}{-(x-1)} = -(2x + 3),$$ and $\lim_{x \to 1^-} f(x) = -(2(1) + 3) = -5$. (b) Since the one-sided limits are 5 and −5, they are unequal, so $\lim_{x \to 1} f(x)$ does not exist. (c) As $x \to \infty$, $x > 1$ so $f(x) = 2x + 3 \to \infty$; more precisely $\lim_{x \to \infty} f(x) = \infty$ (the function grows without bound). As $x \to -\infty$, $x < 1$ so $f(x) = -(2x + 3) = -2x - 3 \to \infty$ as well; indeed $\lim_{x \to -\infty} f(x) = \infty$. (If the intended reading is the ratio of leading terms, note $f(x) \approx \pm 2x$ for large $\lvert x \rvert$, so the function is unbounded in both directions.)	9	
SECTION B (10 marks)			

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Question	Solution	Marks	Remarks
8.	**(b) $\frac{p}{6}$ (c) $c = \frac{p}{6}$ (d) differentiable, $g'(0) = \frac{1}{18}$** (a) Rationalise the numerator by multiplying numerator and denominator by $\sqrt{x^2 + px + 9} + 3$: $$g(x) = \frac{(\sqrt{x^2 + px + 9} - 3)(\sqrt{x^2 + px + 9} + 3)}{x(\sqrt{x^2 + px + 9} + 3)} = \frac{(x^2 + px + 9) - 9}{x(\sqrt{x^2 + px + 9} + 3)}.$$ The numerator becomes $x^2 + px = x(x + p)$, so for $x \neq 0$, $$g(x) = \frac{x(x + p)}{x(\sqrt{x^2 + px + 9} + 3)} = \frac{x + p}{\sqrt{x^2 + px + 9} + 3}.$$ (b) As $x \to 0$, $\sqrt{x^2 + px + 9} \to \sqrt{9} = 3$ and $x + p \to p$. By the quotient rule for limits (denominator limit $3 + 3 = 6 \neq 0$), $$\lim_{x \to 0} g(x) = \frac{p}{6}.$$ (c) For continuity at $x = 0$ we need $g(0) = \lim_{x \to 0} g(x)$, i.e. $c = \frac{p}{6}$. (d) Take $p = 0$, so $g(x) = \frac{\sqrt{x^2 + 9} - 3}{x}$ for $x \neq 0$ and $g(0) = \frac{0}{6} = 0$. Using (a) with $p = 0$, $$g(x) = \frac{x}{\sqrt{x^2 + 9} + 3}.$$ The difference quotient at 0 is $$\frac{g(x) - g(0)}{x - 0} = \frac{1}{x} \cdot \frac{x}{\sqrt{x^2 + 9} + 3} = \frac{1}{\sqrt{x^2 + 9} + 3} \quad (x \neq 0).$$ As $x \to 0$, this tends to $\frac{1}{3 + 3} = \frac{1}{18}$, a finite limit. Hence g is differentiable at $x = 0$ with $g'(0) = \frac{1}{18}$.	10	

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