MARKING SCHEME

MATHEMATICS Extended Part Module M2 · Paper

Topic practice: Introduction to e · 2027

This marking scheme has been prepared for markers' reference. It is not a set of model answers. Award method marks where the working is sound. Accept equivalent methods unless the question specifies otherwise.

Question	Solution	Marks	Remarks
SECTION A(1) (16 marks)			
1.	e^3 Recall the standard limit $\lim_{n\to\infty}(1+\frac{x}{n})^n = e^x$ for any real constant x. Put $x = 3$: $$\lim_{n\to\infty}(1+\frac{3}{n})^n = e^3.$$ (Equivalently, write $(1+\frac{3}{n})^n = [(1+\frac{3}{n})^{n/3}]^3 \to e^3$ since $\frac{n}{3}\to\infty$.)	3	
2.	$x = \ln 2$ **or** $x = \ln 3$ Let $u = e^x$, where $u > 0$. The equation becomes $$u^2 - 5u + 6 = 0.$$ Factorise: $(u-2)(u-3) = 0$, so $u = 2$ or $u = 3$. Both are positive, so both are admissible. Hence $e^x = 2$ gives $x = \ln 2$, and $e^x = 3$ gives $x = \ln 3$.	3	
3.	**Expansion shown; each term tends to $\frac{1}{k!}$, giving $e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots$** By the binomial theorem, $$(1+\frac{1}{n})^n = \sum_{k=0}^{n}\binom{n}{k}\frac{1}{n^k}.$$ Now $\binom{n}{k}\frac{1}{n^k} = \frac{n(n-1)\cdots(n-k+1)}{k!\,n^k} = \frac{1}{k!}(1-\frac{1}{n})(1-\frac{2}{n})\cdots(1-\frac{k-1}{n})$. For $k = 0, 1, 2, 3$ this gives 1, 1, $\frac{n-1}{2n}$, $\frac{(n-1)(n-2)}{6n^2}$, so $$(1+\frac{1}{n})^n = 2 + \frac{n-1}{2n} + \frac{(n-1)(n-2)}{6n^2} + \cdots$$ For fixed k, as $n \to \infty$ each factor $(1-\frac{j}{n}) \to 1$, so the k-th term tends to $\frac{1}{k!}$. Hence $$\lim_{n\to\infty}(1+\frac{1}{n})^n = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots = e.$$ As n increases each factor $(1-\frac{j}{n})$ increases, so every term increases; the sequence is increasing and bounded above by $1 + 1 + \frac{1}{2} + \frac{1}{4} + \cdots = 3$, hence it converges — to e.	4	

TUTGATHER

Question	Solution	Marks	Remarks
4.	**(a)** $f(x) = \ln(1+x) - \ln(1-x)$**; (b) shown** (a) Since $\ln\frac{a}{b} = \ln a - \ln b$ for $a, b > 0$, and $1 + x > 0$, $1 - x > 0$ on $-1 < x < 1$, $f(x) = \ln(1+x) - \ln(1-x)$. (b) Put $x = \frac{2t}{1+t^2}$ with $0 < t < 1$; then $0 < x < 1$, so the formula applies. Compute $$1 + x = 1 + \frac{2t}{1+t^2} = \frac{1+2t+t^2}{1+t^2} = \frac{(1+t)^2}{1+t^2},$$ $$1 - x = 1 - \frac{2t}{1+t^2} = \frac{1-2t+t^2}{1+t^2} = \frac{(1-t)^2}{1+t^2}.$$ Therefore $$\frac{1+x}{1-x} = \frac{(1+t)^2}{(1-t)^2} = \left(\frac{1+t}{1-t}\right)^2.$$ Hence $$f\left(\frac{2t}{1+t^2}\right) = \ln\left(\frac{1+t}{1-t}\right)^2 = 2\ln\left(\frac{1+t}{1-t}\right) = 2f(t).$$	6	
SECTION A(2) (24 marks)			
5.	**(a)** $10000\left(1+\frac{0.06}{n}\right)^n$**, limit** $\$10000e^{0.06} \approx \$10618$**; (b)** $n \geq 2$**; (c)** ≈ 11.6 **years** (a) After 1 year with n compoundings the amount is $$A_n = 10000\left(1+\frac{0.06}{n}\right)^n.$$ As $n \to \infty$, using $\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x$ with $x = 0.06$, $A \to 10000\,e^{0.06} \approx 10000(1.061836\ldots) \approx \$10\,618$. (b) We need $10000\left(1+\frac{0.06}{n}\right)^n > 10600$, i.e. $\left(1+\frac{0.06}{n}\right)^n > 1.06$. For $n = 1$: 1.06 is not greater than 1.06 (equality), so $n = 1$ fails. For $n = 2$: $(1.03)^2 = 1.0609 > 1.06$. Since A_n increases with n, all $n \geq 2$ work. Hence the least number of compoundings is $n = 2$ (i.e. half-yearly). (c) With continuous compounding, $10000\,e^{0.06t} = 20000$, so $e^{0.06t} = 2$ and $0.06t = \ln 2 \Rightarrow t = \frac{\ln 2}{0.06} \approx \frac{0.693147}{0.06} \approx 11.6$ years.	7	

TUTGATHER

Question	Solution	Marks	Remarks
6.	**(a)** $f'(x) = e^x(\sin x + \cos x)$**,** $f''(x) = 2e^x\cos x$**; (b)** $x = \frac{3\pi}{4} + k\pi$**,** $k = 0, 1, 2, \ldots$ **;** **(c)** $x = \frac{3\pi}{4}$**,** $f \approx 7.46$ (a) By the product rule, $$f'(x) = e^x\sin x + e^x\cos x = e^x(\sin x + \cos x).$$ Differentiating again, $$f''(x) = e^x(\sin x + \cos x) + e^x(\cos x - \sin x) = 2e^x\cos x.$$ (b) Since $e^x > 0$, $f'(x) = 0 \iff \sin x + \cos x = 0 \iff \tan x = -1$ (as $\cos x \neq 0$). For $x \geq 0$ this gives $x = \frac{3\pi}{4} + k\pi$, $k = 0, 1, 2, \ldots$ (c) At these points $f''(\frac{3\pi}{4} + k\pi) = 2e^x\cos(\frac{3\pi}{4} + k\pi)$. For k even, $\cos\frac{3\pi}{4} = -\frac{\sqrt{2}}{2} < 0$: local maximum. For k odd, $\cos\frac{7\pi}{4} = \frac{\sqrt{2}}{2} > 0$: local minimum. The smallest $x > 0$ giving a maximum is $x = \frac{3\pi}{4}$. Then $$f(\frac{3\pi}{4}) = e^{3\pi/4}\sin\frac{3\pi}{4} = e^{3\pi/4}\cdot\frac{\sqrt{2}}{2} \approx 10.5507 \times 0.7071 \approx 7.46.$$	8	
7.	**(a)** $P(0) = 10$ **thousand, limit** 50 **thousand; (b)** $k = 0.8$**; (c)** $t = \frac{\ln 4}{0.8} \approx 1.73$ **h,** $P = 25$ **thousand** (a) $P(0) = \frac{50}{1+4} = \frac{50}{5} = 10$ (thousand). As $t \to \infty$, $e^{-0.8t} \to 0$, so $P \to \frac{50}{1} = 50$ (thousand). (b) Write $P = 50(1 + 4e^{-0.8t})^{-1}$. Then $$P' = 50\cdot(-1)(1+4e^{-0.8t})^{-2}\cdot 4(-0.8)e^{-0.8t} = \frac{160\,e^{-0.8t}}{(1+4e^{-0.8t})^2}.$$ Now $1 - \frac{P}{50} = 1 - \frac{1}{1+4e^{-0.8t}} = \frac{4e^{-0.8t}}{1+4e^{-0.8t}}$, so $$kP(1-\frac{P}{50}) = k\cdot\frac{50}{1+4e^{-0.8t}}\cdot\frac{4e^{-0.8t}}{1+4e^{-0.8t}} = \frac{200k\,e^{-0.8t}}{(1+4e^{-0.8t})^2}.$$ Comparing with P', $200k = 160$, so $k = 0.8$. (c) Growth is fastest when P' is maximal, i.e. when $P(1 - \frac{P}{50})$ is maximal. This quadratic in P peaks at $P = 25$. Set $\frac{50}{1+4e^{-0.8t}} = 25 \Rightarrow 1 + 4e^{-0.8t} = 2 \Rightarrow e^{-0.8t} = \frac{1}{4} \Rightarrow t = \frac{\ln 4}{0.8} \approx \frac{1.3863}{0.8} \approx 1.73$ hours. At that instant $P = 25$ thousand.	9	
SECTION B (10 marks)			

TUTGATHER

Question	Solution	Marks	Remarks
8.	**(a) shown; (b)** $g'' \geq 0$ **on** $x \geq 0$**,** g' **increasing with** $g'(0) = 0$ **so** $g' \geq 0$**,** g **increasing with** $g(0) = 0$ **so** $g \geq 0$**; (c) divide by** $x > 0$**; (d) limit** $= 1$ (a) Differentiate term by term: $g'(x) = e^x - 1 - x, \qquad g''(x) = e^x - 1.$ (b) For $x \geq 0$, $e^x \geq 1$, so $g''(x) \geq 0$. Hence g' is increasing on $[0, \infty)$. Since $g'(0) = e^0 - 1 - 0 = 0$, we get $g'(x) \geq 0$ for all $x \geq 0$. Hence g is increasing on $[0, \infty)$. Since $g(0) = 1 - 1 - 0 - 0 = 0$, we get $g(x) \geq 0$ for all $x \geq 0$. Therefore $e^x - 1 - x - \frac{x^2}{2} \geq 0$, i.e. $e^x \geq 1 + x + \frac{x^2}{2}$ for $x \geq 0$. (c) For $x > 0$, subtract 1 from the inequality in (b): $e^x - 1 \geq x + \frac{x^2}{2}$. Dividing by $x > 0$, $\frac{e^x - 1}{x} \geq 1 + \frac{x}{2}.$ Since $x > 0$ the inequality is in fact strict (as $g(x) > 0$ for $x > 0$), giving $\frac{e^x - 1}{x} > 1 + \frac{x}{2}$. (d) Put $x = \frac{1}{n}$ $(n \geq 1)$: $\frac{e^{1/n} - 1}{1/n} > 1 + \frac{1}{2n} \Rightarrow n(e^{1/n} - 1) > 1 + \frac{1}{2n} > 1.$ Also, from the standard limit $\lim_{h \to 0} \frac{e^h - 1}{h} = 1$ (with $h = \frac{1}{n} \to 0$), $\lim_{n \to \infty} n(e^{1/n} - 1) = \lim_{h \to 0} \frac{e^h - 1}{h} = 1.$ This is consistent with the expansion $e^x = 1 + x + \frac{x^2}{2} + \cdots$ near $x = 0$: for small x, $\frac{e^x - 1}{x} \approx 1 + \frac{x}{2}$, so $n(e^{1/n} - 1) \approx 1 + \frac{1}{2n} \to 1$ as $n \to \infty$.	10	

TUTGATHER