MARKING SCHEME

MATHEMATICS Extended Part Module M2 · Paper

Topic practice: The Binomial Theorem · 2027

This marking scheme has been prepared for markers' reference. It is not a set of model answers. Award method marks where the working is sound. Accept equivalent methods unless the question specifies otherwise.

Question	Solution	Marks	Remarks
SECTION A(1) (16 marks)			
1.	$16 - 96x + 216x^2 - 216x^3 + 81x^4$ By the binomial theorem, $$(2-3x)^4 = \sum_{k=0}^{4} \binom{4}{k} 2^{4-k}(-3x)^k.$$ Compute term by term: - $k=0$: $\binom{4}{0}2^4 = 16$; - $k=1$: $\binom{4}{1}2^3(-3x) = 4\cdot 8\cdot(-3)x = -96x$; - $k=2$: $\binom{4}{2}2^2(-3x)^2 = 6\cdot 4\cdot 9x^2 = 216x^2$; - $k=3$: $\binom{4}{3}2^1(-3x)^3 = 4\cdot 2\cdot(-27)x^3 = -216x^3$; - $k=4$: $\binom{4}{4}(-3x)^4 = 81x^4$. Hence $(2-3x)^4 = 16 - 96x + 216x^2 - 216x^3 + 81x^4$.	3	
2.	$\binom{12}{4} = 495$ Using Pascal's identity $\binom{n}{r-1} + \binom{n}{r} = \binom{n+1}{r}$ with $n=11$, $r=4$: $$\binom{11}{3} + \binom{11}{4} = \binom{12}{4}.$$ Check: $\binom{11}{3} = 165$ and $\binom{11}{4} = 330$, so the sum is 495, and $\binom{12}{4} = \frac{12\cdot 11\cdot 10\cdot 9}{4!} = 495$.	3	
3.	-160 The general term is $$\binom{6}{k}(2x)^{6-k}\left(-\frac{1}{x}\right)^k = \binom{6}{k}2^{6-k}(-1)^k x^{6-2k}.$$ For the constant term we need the power of x to be zero: $6-2k=0$, so $k=3$. Thus the constant term is $$\binom{6}{3}2^3(-1)^3 = 20\cdot 8\cdot(-1) = -160.$$	4	

TUTGATHER

Question	Solution	Marks	Remarks
4.	**(a)** $1 + 10x + 40x^2 + 80x^3 + \cdots$ **(b)** 45 (a) By the binomial theorem, $(1+2x)^5 = 1 + \binom{5}{1}(2x) + \binom{5}{2}(2x)^2 + \binom{5}{3}(2x)^3 + \cdots$ $= 1 + 5(2x) + 10(4x^2) + 10(8x^3) + \cdots = 1 + 10x + 40x^2 + 80x^3 + \cdots$ (b) Write $(1-x)^2 = 1 - 2x + x^2$. The coefficient of x^3 in the product is obtained from $(80x^3)(1) + (40x^2)(-2x) + (10x)(x^2) = 80x^3 - 80x^3 + 10x^3 = 10x^3$. Wait — recompute carefully: contributions to x^3 are $80\cdot 1$, $40\cdot(-2)$, $10\cdot 1$, giving $80 - 80 + 10 = 10$. Correction of the arithmetic: the terms of $(1+2x)^5$ are $1,\ 10x,\ 40x^2,\ 80x^3$ and the terms of $(1-x)^2$ are $1,\ -2x,\ x^2$. Pairing powers that sum to 3: $80(1) + 40(-2) + 10(1) = 80 - 80 + 10 = 10$. Hence the coefficient of x^3 is 10. (Note: the answer stated is 10.)	6	
SECTION A(2) (24 marks)			
5.	**(a)** $n = 9$ **(b)** -6 (a) The general term of $(1+ax)^n$ is $\binom{n}{k}a^k x^k$. Hence $\binom{n}{2}a^2 = 54, \qquad \binom{n}{3}a^3 = -108.$ Dividing the second equation by the first: $\dfrac{\binom{n}{3}a^3}{\binom{n}{2}a^2} = \dfrac{-108}{54} = -2.$ Now $\dfrac{\binom{n}{3}}{\binom{n}{2}} = \dfrac{n-2}{3}$, so $\dfrac{n-2}{3}a = -2 \quad \Rightarrow \quad a = \dfrac{-6}{n-2}.$ Substituting into $\binom{n}{2}a^2 = 54$: $\dfrac{n(n-1)}{2}\cdot\dfrac{36}{(n-2)^2} = 54 \Rightarrow 36n(n-1) = 108(n-2)^2 \Rightarrow n(n-1) = 3(n-2)^2.$ Expanding: $n^2 - n = 3n^2 - 12n + 12$, i.e. $2n^2 - 11n + 12 = 0$, so $(2n-3)(n-4) = 0$. Since $n \geq 3$ is an integer, $n = 4$. Check: with $n = 4$, $a = \dfrac{-6}{4-2} = -3$. Then $\binom{4}{2}a^2 = 6\cdot 9 = 54$ ✓ and $\binom{4}{3}a^3 = 4\cdot(-27) = -108$ ✓. So $a = -3$ and $n = 4$. (Note: the values are $a = -3$, $n = 4$.) (b) The coefficient of x is $\binom{4}{1}a = 4(-3) = -12$. (Note: the coefficient of x is -12.)	7	

TUTGATHER

Question	Solution	Marks	Remarks
6.	**(a) proof (b)** 0 **(c)** 512 (a) By the binomial theorem, $$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k .$$ Putting $x = 1$ gives $$2^n = (1+1)^n = \sum_{k=0}^{n} \binom{n}{k}.$$ (b) Put $x = -1$ in the expansion: $$(1-1)^{10} = \sum_{k=0}^{10} \binom{10}{k}(-1)^k = \binom{10}{0} - \binom{10}{1} + \binom{10}{2} - \cdots + \binom{10}{10}.$$ Since $(1-1)^{10} = 0$, the required value is 0. (c) Let $E = \binom{10}{0} + \binom{10}{2} + \cdots + \binom{10}{10}$ (even-indexed terms) and $O = \binom{10}{1} + \binom{10}{3} + \cdots + \binom{10}{9}$ (odd-indexed terms). From (a) with $n = 10$: $E + O = 2^{10} = 1024$. From (b): $E - O = 0$, so $E = O$. Hence $2E = 1024$ and $E = 512$. Therefore $\binom{10}{0} + \binom{10}{2} + \cdots + \binom{10}{10} = 512$.	8	

TUTGATHER

Question	Solution	Marks	Remarks
7.	**(a)** $\binom{n}{3}2^{n-3}\frac{x^3}{8}$ **and** $\binom{n}{4}2^{n-4}\frac{x^4}{16}$ **(b)** $n = 10$ **(c)** -640 (a) The general term is $\binom{n}{k}2^{n-k}\left(\frac{x}{2}\right)^k = \binom{n}{k}2^{n-k}\frac{x^k}{2^k} = \binom{n}{k}2^{n-2k}x^k$. So the term in x^3 is $\binom{n}{3}2^{n-6}x^3$ and the term in x^4 is $\binom{n}{4}2^{n-8}x^4$. (b) The ratio of coefficients is $$\frac{\binom{n}{4}2^{n-8}}{\binom{n}{3}2^{n-6}} = \frac{\binom{n}{4}}{\binom{n}{3}}\cdot\frac{1}{4} = \frac{n-3}{4}\cdot\frac{1}{4} = \frac{n-3}{16}.$$ Given this equals $\frac{3}{2}$: $$\frac{n-3}{16} = \frac{3}{2} \Rightarrow n-3 = 24 \Rightarrow n = 27.$$ Check: $\binom{27}{4}/\binom{27}{3} = \frac{24}{4} = 6$, and $2^{-2} = \frac{1}{4}$, giving $\frac{6}{4} = \frac{3}{2}$ √. So $n = 27$. (Note: the value is $n = 27$.) (c) With $n = 27$, the general term of $\left(2+\frac{x}{2}\right)^{27}$ is $\binom{27}{k}2^{27-2k}x^k$. Now $\left(1-\frac{2}{x}\right)^3 = 1-\frac{6}{x}+\frac{12}{x^2}-\frac{8}{x^3}$. The term independent of x arises from pairing x^k with x^{-j} where $k = j$, $j = 0, 1, 2, 3$: $$\binom{27}{0}2^{27}(-8)+\binom{27}{1}2^{25}(12)+\binom{27}{2}2^{23}(-6)+\binom{27}{3}2^{21}(1).$$ Compute: $2^{27} = 134217728$, $2^{25} = 33554432$, $2^{23} = 8388608$, $2^{21} = 2097152$. - $1\cdot 134217728\cdot(-8) = -1073741824$; - $27\cdot 33554432\cdot 12 = 10871635968$; - $351\cdot 8388608\cdot(-6) = -17664024576$; - $2925\cdot 2097152 = 6134169600$. Sum: $-1073741824 + 10871635968 - 17664024576 + 6134169600 = -1728008832$. Hmm, this is not a clean value; recheck the pairing. The constant term requires $k - j = 0$, i.e. $k = j$; the computation above is correct in structure. Let us recompute the last two terms. $\binom{27}{2} = 351$, $351\cdot 8388608 = 2944401408$, times -6 gives -17666408448. $\binom{27}{3} = 2925$, $2925\cdot 2097152 = 6134169600$. Sum: $-1073741824 + 10871635968 - 17666408448 + 6134169600 = -1384120704$. This is still not clean, so the intended reading is the constant term of $\left(2+\frac{x}{2}\right)^{27}\left(1-\frac{2}{x}\right)^3$ computed directly; the exact value is -1384120704. For a clean answer, note $2^{21} = 2097152$ and factor: $2^{21}[-512 + 27\cdot 16\cdot 12 - 351\cdot 4\cdot 6 + 2925] = 2^{21}[-512 + 5184 - 8424 + 2925] = 2^{21}(-827)$. So the constant term is $-827\cdot 2^{21} = -1734344704$. (Note: the constant term is $-827\cdot 2^{21}$.)	9	
SECTION B (10 marks)			

TUTGATHER

Question	Solution	Marks	Remarks
8.	**(a) proof (b) proof (c)** $n(n+1)2^{n-2}$**; for** $n = 8$**:** 9216 (a) Differentiate both sides of $$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$ with respect to x: $$n(1+x)^{n-1} = \sum_{k=1}^{n} k\binom{n}{k} x^{k-1}.$$ Putting $x = 1$: $$n\,2^{n-1} = \sum_{k=1}^{n} k\binom{n}{k} = \sum_{k=0}^{n} k\binom{n}{k} = T.$$ (b) Differentiate $n(1+x)^{n-1} = \sum_{k=1}^{n} k\binom{n}{k} x^{k-1}$ again: $$n(n-1)(1+x)^{n-2} = \sum_{k=2}^{n} k(k-1)\binom{n}{k} x^{k-2}.$$ Putting $x = 1$: $$n(n-1)2^{n-2} = \sum_{k=2}^{n} k(k-1)\binom{n}{k} = \sum_{k=0}^{n} k(k-1)\binom{n}{k} = U,$$ since the $k = 0$ and $k = 1$ terms vanish. (c) Since $k^2 = k(k-1) + k$, $$\sum_{k=0}^{n} k^2\binom{n}{k} = U + T = n(n-1)2^{n-2} + n\,2^{n-1}.$$ Factor out $n\,2^{n-2}$: $$= n\,2^{n-2}[(n-1) + 2] = n(n+1)2^{n-2}.$$ When $n = 8$: $$8 \cdot 9 \cdot 2^6 = 72 \cdot 64 = 4608.$$ Check directly: $\sum k^2\binom{8}{k}$; using $n(n+1)2^{n-2} = 8 \cdot 9 \cdot 64 = 4608$. (Note: the value is 4608.)	10	

TUTGATHER