MARKING SCHEME

MATHEMATICS Extended Part Module M1 · Paper

Topic practice: Definite Integration · 2027

This marking scheme has been prepared for markers' reference. It is not a set of model answers. Award method marks where the working is sound. Accept equivalent methods unless the question specifies otherwise.

Question	Solution	Marks	Remarks
SECTION A(1) (16 marks)			
1.	$\frac{33}{2}$ Write $3\sqrt{x} = 3x^{1/2}$ and $\frac{2}{x^2} = 2x^{-2}$. An antiderivative is $3\cdot\frac{x^{3/2}}{3/2} - 2\cdot\frac{x^{-1}}{-1} = 2x^{3/2} + \frac{2}{x}$. Hence $\int_1^4 (3\sqrt{x} - \frac{2}{x^2})dx = [2x^{3/2} + \frac{2}{x}]_1^4$. At $x = 4$: $2(4)^{3/2} + \frac{2}{4} = 2(8) + \frac{1}{2} = \frac{33}{2}$. At $x = 1$: $2(1) + \frac{2}{1} = 4$. Therefore the integral equals $\frac{33}{2} - 4 = \frac{25}{2}$... recomputing: $\frac{33}{2} - \frac{8}{2} = \frac{25}{2}$. So the value is $\frac{25}{2}$.	3	
2.	**(a)** 7 **(b)** 46 (a) By the additive property of definite integrals, $\int_0^5 f(x)\,dx = \int_0^2 f(x)\,dx + \int_2^5 f(x)\,dx$. So $12 = 5 + \int_2^5 f(x)\,dx$, giving $\int_2^5 f(x)\,dx = 7$. (b) By linearity, $\int_0^5 (3f(x) + 2)dx = 3\int_0^5 f(x)\,dx + \int_0^5 2\,dx = 3(12) + [2x]_0^5 = 36 + 10 = 46$.	3	
3.	$\frac{1365}{4}$ Let $u = x^2 + 3$, so $\frac{du}{dx} = 2x$ and $x\,dx = \frac{du}{2}$. When $x = 0$, $u = 3$; when $x = 1$, $u = 4$. Hence $\int_0^1 x(x^2 + 3)^5\,dx = \int_3^4 u^5\cdot\frac{du}{2} = \frac{1}{2}[\frac{u^6}{6}]_3^4 = \frac{1}{12}(4^6 - 3^6)$. Now $4^6 = 4096$ and $3^6 = 729$, so the value is $\frac{4096 - 729}{12} = \frac{3367}{12}$. (Check: $3367 = 7\times 481$, so the fraction does not simplify.) Thus the integral equals $\frac{3367}{12}$.	4	

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Question	Solution	Marks	Remarks
4.	**(a)** $\frac{4}{3}$ **(b)** $\frac{8}{3}$ (a) Between $x = 0$ and $x = 2$ the curve lies below the x-axis (e.g. at $x = 1$, $y = -1$). So the area is $$-\int_0^2 (x^2 - 2x)\,dx = -\left[\frac{x^3}{3} - x^2\right]_0^2 = -\left(\frac{8}{3} - 4\right) = \frac{4}{3}.$$ (b) On $[2, 3]$ the curve lies above the x-axis. The region between $x = 2$ and $x = 3$ has area $$\int_2^3 (x^2 - 2x)\,dx = \left[\frac{x^3}{3} - x^2\right]_2^3 = (9 - 9) - \left(\frac{8}{3} - 4\right) = \frac{4}{3}.$$ The region from $x = 0$ to $x = 2$ (below the axis) has area $\frac{4}{3}$ and the region from $x = 2$ to $x = 3$ (above the axis) has area $\frac{4}{3}$. Hence the total area bounded by the curve, the x-axis and $x = 3$ is $$\frac{4}{3} + \frac{4}{3} = \frac{8}{3}.$$	6	
SECTION A(2) (24 marks)			
5.	**(a) shown,** $I = \ln 2$ **(b)** $\frac{1}{2}$ (a) Let $u = 1 + \tan x$. Then $\frac{du}{dx} = \sec^2 x$, so $\sec^2 x\,dx = du$. When $x = 0$, $u = 1 + 0 = 1$; when $x = \frac{\pi}{4}$, $u = 1 + 1 = 2$. Hence $$I = \int_1^2 \frac{du}{u} = [\ln u]_1^2 = \ln 2 - \ln 1 = \ln 2.$$ (b) With the same substitution $u = 1 + \tan x$, $$\int_0^{\pi/4} \frac{\sec^2 x}{(1 + \tan x)^2}dx = \int_1^2 \frac{du}{u^2} = \left[-\frac{1}{u}\right]_1^2 = -\frac{1}{2} + 1 = \frac{1}{2}.$$	7	
6.	**(a)** 28 **(b) shown (c)** 12**;** $\int_0^2 f(2 - x)\,dx = 6$ (a) By linearity, $$\int_0^2 (x + 3)f(x)\,dx = \int_0^2 xf(x)\,dx + 3\int_0^2 f(x)\,dx = 10 + 3(6) = 28.$$ (b) Let $x = 2 - t$. Then $dx = -dt$. When $x = 0$, $t = 2$; when $x = 2$, $t = 0$. Hence $$\int_0^2 f(2 - x)\,dx = \int_2^0 f(t)(-dt) = \int_0^2 f(t)\,dt = \int_0^2 f(x)\,dx.$$ (c) By linearity and part (b), $$\int_0^2 (f(x) + f(2 - x))dx = \int_0^2 f(x)\,dx + \int_0^2 f(2 - x)\,dx = 6 + 6 = 12.$$ Since the two integrals in the sum are equal (part (b)), each equals $\frac{12}{2} = 6$; in particular $\int_0^2 f(2 - x)\,dx = 6$.	8	

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Question	Solution	Marks	Remarks
7.	**(a)** $-\frac{\ln x+1}{x}+C$ **(b)** $1-\frac{2}{e}$ **(c)** $1-\frac{3}{e^2}$ (a) Let $u=\ln x$, so $du=\frac{dx}{x}$ and $x=e^u$, giving $\frac{1}{x^2}=e^{-2u}$. Then $$\int \frac{\ln x}{x^2}dx=\int u\,e^{-2u}\,du.$$ Integrating by parts with $p=u$, $dq=e^{-2u}du$: $dp=du$, $q=-\frac{e^{-2u}}{2}$, so $$\int ue^{-2u}du=-\frac{ue^{-2u}}{2}+\frac{1}{2}\int e^{-2u}du=-\frac{ue^{-2u}}{2}-\frac{e^{-2u}}{4}+C.$$ Back-substituting $u=\ln x$, $e^{-2u}=\frac{1}{x^2}$: $$\int \frac{\ln x}{x^2}dx=-\frac{\ln x}{2x^2}-\frac{1}{4x^2}+C=-\frac{2\ln x+1}{4x^2}+C.$$ (Equivalently, one checks that $\frac{d}{dx}\left(-\frac{\ln x+1}{x}\right)=\frac{\ln x}{x^2}$.) (b) On $[1,e]$, $\ln x\geq 0$, so the curve is above the x-axis. The area is $$\int_1^e \frac{\ln x}{x^2}dx=\left[-\frac{\ln x+1}{x}\right]_1^e.$$ At $x=e$: $-\frac{1+1}{e}=-\frac{2}{e}$. At $x=1$: $-\frac{0+1}{1}=-1$. Hence the area is $-\frac{2}{e}+1=1-\frac{2}{e}$. (c) On $[e,e^2]$, $\ln x>0$ again, so the area is $$\int_e^{e^2} \frac{\ln x}{x^2}dx=\left[-\frac{\ln x+1}{x}\right]_e^{e^2}.$$ At $x=e^2$: $-\frac{2+1}{e^2}=-\frac{3}{e^2}$. At $x=e$: $-\frac{2}{e}$. Hence the area is $-\frac{3}{e^2}+\frac{2}{e}=\frac{2}{e}-\frac{3}{e^2}$. (Note: the region from $x=1$ to $x=e^2$ has total area $(1-\frac{2}{e})+(\frac{2}{e}-\frac{3}{e^2})=1-\frac{3}{e^2}$.)	9	
SECTION B (10 marks)			

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Question	Solution	Marks	Remarks
8.	**(a)** $f(x) = x(x-1)(x-2)$**; intercepts** $x = 0, 1, 2$ **(b) local max** $(1 - \frac{\sqrt{3}}{3}, \frac{2\sqrt{3}}{9})$**, local min** $(1 + \frac{\sqrt{3}}{3}, -\frac{2\sqrt{3}}{9})$ **(c)** $\frac{1}{2}$ **(d)** $k = 1$ (a) $f(x) = x(x^2 - 3x + 2) = x(x-1)(x-2)$. The x-intercepts are $x = 0$, $x = 1$, $x = 2$. (b) $f'(x) = 3x^2 - 6x + 2$. Setting $f'(x) = 0$: $x = \frac{6 \pm \sqrt{36 - 24}}{6} = 1 \pm \frac{\sqrt{3}}{3}$. Since $f''(x) = 6x - 6$, at $x = 1 - \frac{\sqrt{3}}{3}$ we have $f'' < 0$ (local maximum) and at $x = 1 + \frac{\sqrt{3}}{3}$, $f'' > 0$ (local minimum). Using $f(x) = x(x-1)(x-2)$ with $x - 1 = \mp \frac{\sqrt{3}}{3}$ and $x(x-2) = (x-1)^2 - 1 = \frac{1}{3} - 1 = -\frac{2}{3}$: $$f(1 \mp \tfrac{\sqrt{3}}{3}) = (-\tfrac{2}{3})(\mp \tfrac{\sqrt{3}}{3}) = \pm \tfrac{2\sqrt{3}}{9}.$$ So the local maximum is $(1 - \frac{\sqrt{3}}{3}, \frac{2\sqrt{3}}{9})$ and the local minimum is $(1 + \frac{\sqrt{3}}{3}, -\frac{2\sqrt{3}}{9})$. (c) On $(0, 1)$, $f(x) = x(x-1)(x-2) > 0$; on $(1, 2)$, $f(x) < 0$. An antiderivative is $F(x) = \frac{x^4}{4} - x^3 + x^2$. $\int_0^1 f(x)dx = F(1) - F(0) = \frac{1}{4} - 1 + 1 = \frac{1}{4}$, $\int_1^2 f(x)dx = F(2) - F(1) = (4 - 8 + 4) - \frac{1}{4} = -\frac{1}{4}$. Total area $= \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$. (d) The line $y = kx$ meets the curve where $x^3 - 3x^2 + 2x = kx$, i.e. $x(x^2 - 3x + (2 - k)) = 0$. Besides $x = 0$, the other intersections satisfy $x^2 - 3x + (2 - k) = 0$, with roots $x = a$ and $x = b$ where $a + b = 3$ and $ab = 2 - k$. For three distinct points we need $k > -\frac{1}{4}$ and $k \neq 2$. The two enclosed regions lie between $x = 0$ and $x = a$, and between $x = a$ and $x = b$. Their areas are $$A_1 = \int_0^a (f(x) - kx)dx, \qquad A_2 = \left| \int_a^b (f(x) - kx)dx \right|.$$ Let $g(x) = f(x) - kx = x(x^2 - 3x + (2 - k)) = x(x-a)(x-b)$. Since g changes sign at $x = a$ but not at $x = 0$ or $x = b$, the condition $A_1 = A_2$ is equivalent to $$\int_0^b g(x)\,dx = 0.$$ Now $$\int_0^b (x^3 - 3x^2 + (2 - k)x)dx = \frac{b^4}{4} - b^3 + \frac{(2 - k)b^2}{2}.$$ Since b is a root, $b^2 - 3b + (2 - k) = 0$, so $2 - k = 3b - b^2$. Substituting: $$\frac{b^4}{4} - b^3 + \frac{(3b - b^2)b^2}{2} = \frac{b^4}{4} - b^3 + \frac{3b^3}{2} - \frac{b^4}{2} = -\frac{b^4}{4} + \frac{b^3}{2} = \frac{b^3(2 - b)}{4}.$$ Setting this to 0 with $b \neq 0$ gives $b = 2$. Then $a = 3 - b = 1$, and $2 - k = ab = 2$, so $k = 0$. Wait — check the sign condition: with $k = 0$ the line is the x-axis and the two regions are those of part (c), which have equal areas $\frac{1}{4}$ each. Indeed $k = 0$. Re-examining: the required value is $k = 0$; the two enclosed regions (between $x = 0$ and $x = 1$, and between $x = 1$ and $x = 2$) then have equal areas $\frac{1}{4}$. Therefore $k = 0$.	10	

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