MARKING SCHEME

MATHEMATICS Extended Part Module M1 · Paper

Topic practice: Second Derivative · 2027

This marking scheme has been prepared for markers' reference. It is not a set of model answers. Award method marks where the working is sound. Accept equivalent methods unless the question specifies otherwise.

Question	Solution	Marks	Remarks
SECTION A(1) (16 marks)			
1.	$\frac{d^2y}{dx^2} = 36x^2 - 10$ Differentiate once: $\frac{dy}{dx} = 12x^3 - 10x + 7.$ Differentiate again: $\frac{d^2y}{dx^2} = 36x^2 - 10.$	3	
2.	$x = 2$ **(the only point of inflection is** $(2,\ 3)$**)** $\frac{dy}{dx} = 3x^2 - 12x + 9, \qquad \frac{d^2y}{dx^2} = 6x - 12.$ Set $\frac{d^2y}{dx^2} = 0$: $6x - 12 = 0 \Rightarrow x = 2$. Check the sign change of $\frac{d^2y}{dx^2}$: for $x < 2$ it is negative, for $x > 2$ it is positive. Since the sign changes, $x = 2$ gives a point of inflection. When $x = 2$, $y = 8 - 24 + 18 + 1 = 3$, so the point of inflection is $(2,\ 3)$.	3	
3.	$\frac{d^2y}{dx^2} = e^{-x}(x^2 - 4x + 2)$**; at** $x = 0$ **it equals** 2**.** By the product rule, $\frac{dy}{dx} = 2xe^{-x} + x^2(-e^{-x}) = e^{-x}(2x - x^2).$ Differentiate again using the product rule: $\frac{d^2y}{dx^2} = -e^{-x}(2x - x^2) + e^{-x}(2 - 2x) = e^{-x}(x^2 - 2x + 2 - 2x) = e^{-x}(x^2 - 4x + 2).$ At $x = 0$: $\frac{d^2y}{dx^2} = e^0(0 - 0 + 2) = 2.$	4	

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Question	Solution	Marks	Remarks
4.	**(a)** $\frac{d^2y}{dx^2} = 12x^2 - 24x = 12x(x-2)$**; concave upwards on** $x < 0$ **and** $x > 2$**, concave downwards on** $0 < x < 2$**. (b) Points of inflection:** $(0,\ 0)$ **and** $(2,\ -16)$**.** (a) $$\frac{dy}{dx} = 4x^3 - 12x^2, \qquad \frac{d^2y}{dx^2} = 12x^2 - 24x = 12x(x-2).$$ Sign of $\frac{d^2y}{dx^2}$: - $x < 0$: positive, so C is concave upwards; - $0 < x < 2$: negative, so C is concave downwards; - $x > 2$: positive, so C is concave upwards. (b) $\frac{d^2y}{dx^2} = 0$ at $x = 0$ and $x = 2$. From (a) the sign of $\frac{d^2y}{dx^2}$ changes at each of these values, so both give points of inflection. When $x = 0$, $y = 0$; when $x = 2$, $y = 16 - 32 = -16$. Hence the points of inflection are $(0,\ 0)$ and $(2,\ -16)$.	6	
SECTION A(2) (24 marks)			
5.	**(a)** $\frac{dy}{dx} = \frac{x(x-2)}{(x-1)^2}$**,** $\frac{d^2y}{dx^2} = \frac{2}{(x-1)^3}$**. (b) Stationary points at** $x = 0$ **(local maximum,** $y = 0$**) and** $x = 2$ **(local minimum,** $y = 4$**).** (a) By the quotient rule, $$\frac{dy}{dx} = \frac{2x(x-1) - x^2 \cdot 1}{(x-1)^2} = \frac{2x^2 - 2x - x^2}{(x-1)^2} = \frac{x^2 - 2x}{(x-1)^2} = \frac{x(x-2)}{(x-1)^2}.$$ Differentiate again by the quotient rule: $$\frac{d^2y}{dx^2} = \frac{(2x-2)(x-1)^2 - (x^2-2x)\cdot 2(x-1)}{(x-1)^4} = \frac{(2x-2)(x-1) - 2(x^2-2x)}{(x-1)^3}.$$ Numerator: $(2x-2)(x-1) - 2x^2 + 4x = 2x^2 - 4x + 2 - 2x^2 + 4x = 2$. So $\frac{d^2y}{dx^2} = \frac{2}{(x-1)^3}$. (b) $\frac{dy}{dx} = 0 \Rightarrow x = 0$ or $x = 2$. - At $x = 0$: $\frac{d^2y}{dx^2} = \frac{2}{(-1)^3} = -2 < 0$, so $x = 0$ gives a local maximum; $y = 0$. - At $x = 2$: $\frac{d^2y}{dx^2} = \frac{2}{1^3} = 2 > 0$, so $x = 2$ gives a local minimum; $y = \frac{4}{1} = 4$. Hence the curve has a local maximum at $(0,\ 0)$ and a local minimum at $(2,\ 4)$.	7	

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Question	Solution	Marks	Remarks
6.	**(a)** $\frac{dy}{dx} = 3x^2 - 6x$, $\frac{d^2y}{dx^2} = 6x - 6$. **(b)** $(1,\ 2)$. **(c)** $y = -3x + 5$. (a) $$\frac{dy}{dx} = 3x^2 - 6x, \qquad \frac{d^2y}{dx^2} = 6x - 6.$$ (b) $\frac{d^2y}{dx^2} = 0 \Rightarrow x = 1$. For $x < 1$, $\frac{d^2y}{dx^2} < 0$; for $x > 1$, $\frac{d^2y}{dx^2} > 0$. The sign changes, so $x = 1$ gives a point of inflection. When $x = 1$, $y = 1 - 3 + 4 = 2$. The point of inflection is $(1,\ 2)$. (c) At $x = 1$, $\frac{dy}{dx} = 3 - 6 = -3$. The tangent has slope -3 and passes through $(1,\ 2)$: $$y - 2 = -3(x - 1) \Rightarrow y = -3x + 5.$$	8	
7.	**(a)** $f'(x) = x(2\ln x + 1)$, $f''(x) = 2\ln x + 3$. **(b)** $x = e^{-3/2}$**; sign of** f'' **changes from negative to positive, so it is a point of inflection. (c) Concave downwards on** $0 < x < 1$**.** (a) By the product rule, $$f'(x) = 2x\ln x + x^2 \cdot \frac{1}{x} = 2x\ln x + x = x(2\ln x + 1).$$ Differentiate again: $$f''(x) = 2\ln x + 2x \cdot \frac{1}{x} + 1 = 2\ln x + 2 + 1 = 2\ln x + 3.$$ (b) $f''(x) = 0 \Rightarrow 2\ln x + 3 = 0 \Rightarrow \ln x = -\frac{3}{2} \Rightarrow x = e^{-3/2}$. For $0 < x < e^{-3/2}$, $\ln x < -\frac{3}{2}$ so $f''(x) < 0$; for $x > e^{-3/2}$, $f''(x) > 0$. Since f'' changes sign at $x = e^{-3/2}$, this value gives a point of inflection. (c) For $0 < x < 1$, $\ln x < 0$, so $f''(x) = 2\ln x + 3 < 3$; more precisely, at x close to 1, $f''(x) \to 3 > 0$, so we compare with the threshold $e^{-3/2} \approx 0.223$. On $0 < x < e^{-3/2}$, $f''(x) < 0$ (concave downwards); on $e^{-3/2} < x < 1$, $f''(x) > 0$ (concave upwards). Hence f is concave downwards only on $0 < x < e^{-3/2}$ and concave upwards on $e^{-3/2} < x < 1$; on the whole interval $0 < x < 1$ it is not concave downwards throughout.	9	
SECTION B (10 marks)			

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Question	Solution	Marks	Remarks
8.	**(a)** $f''(x) = \dfrac{18(1-x^2)}{(x^2+3)^3}$**. (b) Points of inflection at** $x = -1$ **and** $x = 1$**; concave upwards on** $-1 < x < 1$**, concave downwards on** $x < -1$ **and** $x > 1$**. (c)** $f'(0) = 0$**,** $f''(0) = 2 > 0$**: local minimum at** $(0, 0)$**; the curve is concave upwards near** $x = 0$**.** (a) By the quotient rule, $$f'(x) = \frac{2x(x^2+3) - x^2(2x)}{(x^2+3)^2} = \frac{2x^3 + 6x - 2x^3}{(x^2+3)^2} = \frac{6x}{(x^2+3)^2}.$$ Differentiate again: $$f''(x) = \frac{6(x^2+3)^2 - 6x \cdot 2(x^2+3) \cdot 2x}{(x^2+3)^4} = \frac{6(x^2+3) - 24x^2}{(x^2+3)^3}.$$ Numerator: $6x^2 + 18 - 24x^2 = 18 - 18x^2 = 18(1-x^2)$. So $f''(x) = \dfrac{18(1-x^2)}{(x^2+3)^3}$. (b) Since $(x^2+3)^3 > 0$ for all x, the sign of $f''(x)$ is the sign of $1 - x^2 = (1-x)(1+x)$. - $x < -1$: $1 - x^2 < 0$, concave downwards; - $-1 < x < 1$: $1 - x^2 > 0$, concave upwards; - $x > 1$: $1 - x^2 < 0$, concave downwards. $f''(x) = 0$ at $x = \pm 1$ and the sign changes at each, so both are points of inflection. Their coordinates: $f(\pm 1) = \dfrac{1}{4}$, giving $(-1, \frac{1}{4})$ and $(1, \frac{1}{4})$. (c) $f'(0) = 0$, so $x = 0$ is a stationary point. $f''(0) = \dfrac{18}{27} = \dfrac{2}{3} > 0$, so by the second derivative test $x = 0$ gives a local minimum; $f(0) = 0$, i.e. the local minimum is at $(0, 0)$. Near $x = 0$ the curve lies in the interval $-1 < x < 1$ where it is concave upwards, so it opens upwards around the minimum, rising on both sides towards the points of inflection at $x = \pm 1$, beyond which it becomes concave downwards and flattens towards the horizontal asymptote $y = 1$.	10	

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