MARKING SCHEME

MATHEMATICS Compulsory Part · Paper 1

Topic practice: Inquiry and Investigation · 2027

This marking scheme has been prepared for markers' reference. It is not a set of model answers. Award method marks where the working is sound. Accept equivalent methods unless the question specifies otherwise.

Question	Solution	Marks	Remarks
SECTION A(1) (16 marks)			
1.	2, 2, 2, 2**; the sequence is constant.** Compute: $a_1 = 2$, $a_2 = 3(2) - 4 = 2$, $a_3 = 3(2) - 4 = 2$, $a_4 = 3(2) - 4 = 2$. Thus the first four terms are 2, 2, 2, 2. The recurrence gives $a_{n+1} = a_n$ whenever $a_n = 2$, so the sequence is constant.	3	
2.	**(a)** \$740**; (b) 125 items.** (a) $C = 500 + 12(20) = 500 + 240 = 740$. (b) Solve $500 + 12n \le 2000$: $12n \le 1500$, so $n \le 125$. Hence the maximum number is 125.	3	
3.	**(a)** $A = x(20 - \frac{x}{2})$**; (b)** 200 m^2**.** (a) The two sides perpendicular to the wall each have length $\frac{40 - x}{2} = 20 - \frac{x}{2}$. Thus $A = x(20 - \frac{x}{2}) = 20x - \frac{x^2}{2}$. (b) Complete the square: $A = -\frac{1}{2}(x - 20)^2 + 200$. Maximum area is 200 m^2 when $x = 20$ m.	4	
4.	**(a) For** $n = 1, 2, 3$ **the values are 43, 47, 53, all prime. (b)** $n = 40$ **gives** $40^2 + 40 + 41 = 1681 = 41^2$**, not prime. (c) A general statement requires a logical proof; checking finitely many cases cannot cover infinitely many possibilities.** (a) $n = 1$: $1 + 1 + 41 = 43$ (prime). $n = 2$: $4 + 2 + 41 = 47$ (prime). $n = 3$: $9 + 3 + 41 = 53$ (prime). (b) Take $n = 40$: $40^2 + 40 + 41 = 1600 + 40 + 41 = 1681 = 41^2$, which is composite. (c) Testing only verifies the statement for those particular values; it does not establish it for all positive integers. A proof is needed to cover infinitely many cases.	6	
SECTION A(2) (24 marks)			
5.	**(a)** $u_2 = \frac{1}{3}$, $u_3 = \frac{1}{5}$**. (b)** $u_n = \frac{1}{2n-1}$**. (c) Proof by induction.** (a) $u_2 = \frac{1}{2(1)+1} = \frac{1}{3}$; $u_3 = \frac{1/3}{2(1/3)+1} = \frac{1/3}{5/3} = \frac{1}{5}$. (b) The denominators are 1, 3, 5, … , so conjecture $u_n = \frac{1}{2n-1}$. (c) Base case $n = 1$: $\frac{1}{2(1)-1} = 1 = u_1$. Assume $u_k = \frac{1}{2k-1}$. Then $u_{k+1} = \frac{u_k}{2u_k + 1} = \frac{\frac{1}{2k-1}}{\frac{2}{2k-1} + 1} = \frac{\frac{1}{2k-1}}{\frac{2k+1}{2k-1}} = \frac{1}{2k+1} = \frac{1}{2(k+1)-1}$. Thus the formula holds for $n = k + 1$. By induction, $u_n = \frac{1}{2n-1}$ for all positive integers n.	7	

Question	Solution	Marks	Remarks
6.	**(a)** $P_0 = 2$**,** $k \approx 0.693$**; (b)** $P \approx 63.5$ **thousand; (c)** $t \approx 5.64$ **hours; (d) e.g. unlimited growth, no resource constraints.** (a) At $t = 0$, $P = P_0 = 2$. At $t = 3$, $2e^{3k} = 16$, so $e^{3k} = 8$, $3k = \ln 8$, $k = \frac{\ln 8}{3} \approx 0.6931$. (b) $P(5) = 2e^{5k} = 2e^{5\ln 8/3} = 2 \cdot 8^{5/3} = 2 \cdot 32 = 64$ thousand (exact). (c) Solve $2e^{kt} = 100$: $e^{kt} = 50$, $t = \frac{\ln 50}{k} \approx \frac{3.912}{0.6931} \approx 5.64$ hours. (d) The model assumes unlimited growth; in reality resources are limited, so growth would slow down.	8	
7.	**(a)** $h = \frac{100}{\pi r^2}$**; (b) shown; (c)** $r \approx 2.20$ **m, minimum cost** $\approx \$2730$**.** (a) $\pi r^2 h = 100$, so $h = \frac{100}{\pi r^2}$. (b) Curved surface area $= 2\pi rh = 2\pi r \cdot \frac{100}{\pi r^2} = \frac{200}{r}$. Cost $= 20 \cdot \frac{200}{r} = \frac{4000}{r}$. Top and bottom area $= 2\pi r^2$, cost $= 30 \cdot 2\pi r^2 = 60\pi r^2$. Total $C = 60\pi r^2 + \frac{4000}{r}$. (c) $\frac{dC}{dr} = 120\pi r - \frac{4000}{r^2}$. Set to 0: $120\pi r^3 = 4000$, $r^3 = \frac{100}{3\pi}$, $r = (\frac{100}{3\pi})^{1/3} \approx 2.197$ m. $\frac{d^2C}{dr^2} = 120\pi + \frac{8000}{r^3} > 0$, so minimum. $C_{\min} = 60\pi(2.197)^2 + \frac{4000}{2.197} \approx 910 + 1821 \approx 2731$ dollars, i.e. \$2730 (3 s.f.).	9	
SECTION B (10 marks)			
8.	**(a)** $x_2 = \frac{3}{2}$**,** $x_3 = \frac{7}{5}$**. (b) Proof by induction. (c) Algebraic verification. (d) Proof using (b) and (c). (e) Decreasing for** $n \geq 2$**.** (a) $x_2 = \frac{1+2}{1+1} = \frac{3}{2}$; $x_3 = \frac{3/2+2}{3/2+1} = \frac{7/2}{5/2} = \frac{7}{5}$. (b) Base case $n = 1$: $x_1 = 1 > 0$. Assume $x_k > 0$. Then $x_{k+1} = \frac{x_k+2}{x_k+1} > 0$ since numerator and denominator are positive. Thus $x_n > 0$ for all n. (c) $x_{n+1} - \sqrt{2} = \frac{x_n+2}{x_n+1} - \sqrt{2} = \frac{x_n+2-\sqrt{2}x_n-\sqrt{2}}{x_n+1} = \frac{(1-\sqrt{2})x_n+(2-\sqrt{2})}{x_n+1} = \frac{(1-\sqrt{2})(x_n-\sqrt{2})}{x_n+1}$. (d) For $n \geq 2$, $x_n > 0$ by (b), so $x_n + 1 > 0$. Also $1 - \sqrt{2} < 0$. From (c), $x_{n+1} - \sqrt{2}$ has the sign of $-(x_n - \sqrt{2})$. For $n = 2$, $x_2 = \frac{3}{2} > \sqrt{2}$ since $\frac{9}{4} > 2$. Assume $x_k > \sqrt{2}$. Then $x_k - \sqrt{2} > 0$, so $x_{k+1} - \sqrt{2} < 0$, i.e. $x_{k+1} < \sqrt{2}$. This alternates, so the statement "$x_n > \sqrt{2}$ for all $n \geq 2$" is false. (e) $x_{n+1} - x_n = \frac{x_n+2}{x_n+1} - x_n = \frac{2-x_n^2}{x_n+1}$. For $n \geq 2$, $x_n > 0$ and $x_n^2 > 2$ (from $x_2 > \sqrt{2}$ and alternation? Actually $x_2 > \sqrt{2}$, $x_3 < \sqrt{2}$; but $x_3 = \frac{7}{5} > \sqrt{2}$? $\frac{7}{5} = 1.4 < 1.414$, so $x_3 < \sqrt{2}$). Thus x_n^2 is not always > 2. The sequence alternates around $\sqrt{2}$ and is not monotonic. However, $x_{n+1} - x_n$ has the sign of $2 - x_n^2$. Since $x_2 > \sqrt{2}$, $x_3 < x_2$; $x_3 < \sqrt{2}$, so $x_4 > x_3$; etc. The sequence is not monotonic; it oscillates and converges to $\sqrt{2}$.	10	

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