MARKING SCHEME

MATHEMATICS Compulsory Part · Paper 1

Topic practice: Equations of Circles · 2027

This marking scheme has been prepared for markers' reference. It is not a set of model answers. Award method marks where the working is sound. Accept equivalent methods unless the question specifies otherwise.

Question	Solution	Marks	Remarks
SECTION A(1) (16 marks)			
1.	$x^2 + y^2 + 4x - 6y - 87 = 0$ The radius r is the distance from the centre $(-2, 3)$ to $(4, -5)$: $r = \sqrt{(4+2)^2 + (-5-3)^2} = \sqrt{36+64} = \sqrt{100} = 10.$ The equation is $(x+2)^2 + (y-3)^2 = 100.$ Expanding: $x^2 + 4x + 4 + y^2 - 6y + 9 = 100$, i.e. $x^2 + y^2 + 4x - 6y - 87 = 0.$	3	
2.	**Centre** $(3, -2)$**, radius** 4 Divide the whole equation by 2: $x^2 + y^2 - 6x + 4y - 3 = 0.$ Complete the squares: $(x^2 - 6x + 9) + (y^2 + 4y + 4) = 3 + 9 + 4,$ $(x-3)^2 + (y+2)^2 = 16.$ Hence the centre is $(3, -2)$ and the radius is $\sqrt{16} = 4$.	3	
3.	$(3, 4)$ **and** $(-4, -3)$ Substitute $y = x + 1$ into $x^2 + y^2 = 25$: $x^2 + (x+1)^2 = 25 \Rightarrow 2x^2 + 2x + 1 = 25 \Rightarrow 2x^2 + 2x - 24 = 0 \Rightarrow x^2 + x - 12 = 0.$ Factorising, $(x-3)(x+4) = 0$, so $x = 3$ or $x = -4$. When $x = 3$, $y = 4$; when $x = -4$, $y = -3$. Hence $A(3, 4)$ and $B(-4, -3)$ (in either order).	4	

TUTGATHER

Question	Solution	Marks	Remarks
4.	**(a) Centre** $(3, 6)$**, radius** $4\sqrt{2}$ **(b)** $x^2 + y^2 - 6x - 12y + 5 = 0$ **(c) Area** $= 32$ (a) The centre is the midpoint of AB: $$(\frac{-1+7}{2}, \frac{2+10}{2}) = (3, 6).$$ The radius is half of AB: $$r = \tfrac{1}{2}\sqrt{(7+1)^2 + (10-2)^2} = \tfrac{1}{2}\sqrt{64+64} = \tfrac{1}{2}\sqrt{128} = 4\sqrt{2}.$$ (b) The equation is $(x-3)^2 + (y-6)^2 = 32$, i.e. $$x^2 + y^2 - 6x - 12y + 9 + 36 - 32 = 0 \Rightarrow x^2 + y^2 - 6x - 12y + 13 = 0.$$ (Check: substituting A: $1 + 4 + 6 - 24 + 13 = 0$.) (c) At $P(7, 2)$: $(7-3)^2 + (2-6)^2 = 16 + 16 = 32$, so P lies on C. Since AB is a diameter, the angle $\angle APB = 90^\circ$ (angle in a semicircle). $$AP = \sqrt{(7+1)^2 + (2-2)^2} = 8, \qquad BP = \sqrt{(7-7)^2 + (2-10)^2} = 8.$$ Area $= \tfrac{1}{2} \times 8 \times 8 = 32$.	6	
SECTION A(2) (24 marks)			
5.	**(b)** $x^2 + y^2 - 6x - 4y + 5 = 0$ **(c)** $x^2 + y^2 - 2x - 2y - 3 = 0$ (a) Since A and B both lie on C and have the same y-coordinate, the perpendicular bisector of AB is the vertical line $x = 3$. The centre of a circle lies on the perpendicular bisector of every chord, so the centre lies on $x = 3$. (b) Let the centre be $(3, k)$. Then $CA = CD$: $$(3-1)^2 + (k-0)^2 = (3-3)^2 + (k-6)^2,$$ $$4 + k^2 = (k-6)^2 = k^2 - 12k + 36.$$ So $12k = 32$, giving $k = \tfrac{8}{3}$. The radius squared is $$r^2 = (3-1)^2 + (\tfrac{8}{3})^2 = 4 + \tfrac{64}{9} = \tfrac{100}{9}.$$ Hence C: $(x-3)^2 + (y-\tfrac{8}{3})^2 = \tfrac{100}{9}$, i.e. $$x^2 + y^2 - 6x - \tfrac{16}{3}y + 9 + \tfrac{64}{9} - \tfrac{100}{9} = 0 \Rightarrow x^2 + y^2 - 6x - \tfrac{16}{3}y + 5 = 0.$$ Multiplying by 3: $3x^2 + 3y^2 - 18x - 16y + 15 = 0$. (Check with $D(3, 6)$: $9 + 36 - 18 - 32 + 5 = 0$.) (c) The centre moves from $(3, \tfrac{8}{3})$ to $(-1, \tfrac{5}{3})$; the radius is unchanged. So $$C' : (x+1)^2 + (y-\tfrac{5}{3})^2 = \tfrac{100}{9}.$$ Expanding and multiplying by 9: $$9(x^2 + 2x + 1) + 9(y^2 - \tfrac{10}{3}y + \tfrac{25}{9}) = 100,$$ $$9x^2 + 9y^2 + 18x - 30y + 9 + 25 - 100 = 0 \Rightarrow 9x^2 + 9y^2 + 18x - 30y - 66 = 0,$$ i.e. $3x^2 + 3y^2 + 6x - 10y - 22 = 0$, or equivalently $x^2 + y^2 + 2x - \tfrac{10}{3}y - \tfrac{22}{3} = 0$.	7	

TUTGATHER

Question	Solution	Marks	Remarks
6.	**(a)** $G(4, -3)$**,** $r = 3$ **(b)(i)** $PT = 4$ **(ii)** $x = 8$ **or** $7x - 24y - 32 = 0$ **(c)** $\sqrt{41} - 3$ (a) Complete the squares: $$(x-4)^2 + (y+3)^2 = 16 + 9 - 16 = 9.$$ So $G(4, -3)$ and $r = 3$. (b)(i) $PG = \sqrt{(8-4)^2 + (1+3)^2} = \sqrt{16+16} = \sqrt{32} = 4\sqrt{2}$. Since $GT \perp PT$ (tangent $\perp$ radius), triangle PGT is right-angled at T: $$PT = \sqrt{PG^2 - r^2} = \sqrt{32 - 9} = \sqrt{23}.$$ (ii) Let the tangent through $P(8, 1)$ have slope m: $y - 1 = m(x - 8)$, i.e. $mx - y + (1 - 8m) = 0$. Its distance from $G(4, -3)$ equals $r = 3$: $$\frac{\lvert 4m + 3 + 1 - 8m \rvert}{\sqrt{m^2+1}} = 3 \Rightarrow \frac{\lvert 4 - 4m \rvert}{\sqrt{m^2+1}} = 3.$$ Squaring: $(4-4m)^2 = 9(m^2+1)$, i.e. $16 - 32m + 16m^2 = 9m^2 + 9$, so $7m^2 - 32m + 7 = 0$. $$m = \frac{32 \pm \sqrt{1024 - 196}}{14} = \frac{32 \pm \sqrt{828}}{14} = \frac{16 \pm 3\sqrt{23}}{7}.$$ The two tangents are $y - 1 = \frac{16 + 3\sqrt{23}}{7}(x - 8)$ and $y - 1 = \frac{16 - 3\sqrt{23}}{7}(x - 8)$. (c) The shortest distance is along PG: $PG - r = 4\sqrt{2} - 3 \approx 2.66$.	8	
7.	**(a) Centre** $(2, -1)$**,** $r = 5$ **(c)** $m < -\frac{3}{4}$ **or** $m > \frac{4}{3}$ **(d)** $\sqrt{46}$ (a) $(x-2)^2 + (y+1)^2 = 20 + 4 + 1 = 25$, so centre $(2, -1)$, radius 5. (b) Substitute $y = mx + 6$: $$x^2 + (mx+6)^2 - 4x + 2(mx+6) - 20 = 0,$$ $$x^2 + m^2x^2 + 12mx + 36 - 4x + 2mx + 12 - 20 = 0,$$ $$(1+m^2)x^2 + (14m - 4)x + 28 = 0.$$ (c) Two distinct intersections $\Leftrightarrow$ discriminant > 0: $$(14m-4)^2 - 4(1+m^2)(28) > 0,$$ $$196m^2 - 112m + 16 - 112 - 112m^2 > 0,$$ $$84m^2 - 112m - 96 > 0 \Rightarrow 3m^2 - 4m - \tfrac{24}{7} > 0.$$ Roots: $m = \frac{4 \pm \sqrt{16 + \frac{288}{7}}}{6} = \frac{4 \pm \sqrt{\frac{400}{7}}}{6} = \frac{4 \pm \frac{20}{\sqrt{7}}}{6}$. Numerically $m \approx -0.593$ or $m \approx 1.926$. Hence L meets C twice when $m < -0.593$ or $m > 1.93$ (3 s.f.). (d) When $m = 1$: $2x^2 + 10x + 28 = 0 \Rightarrow x^2 + 5x + 14 = 0$, whose discriminant is $25 - 56 < 0$. So with $m = 1$ the line misses the circle; the chord length is undefined. (For a valid case, e.g. $m = 2$: $5x^2 + 24x + 28 = 0$, roots $x = -2, -2.8$; chord $= \sqrt{1+4} \cdot 0.8 = 4\sqrt{5}/5 \approx 1.79$.)	9	
SECTION B (10 marks)			

TUTGATHER

Question	Solution	Marks	Remarks
8.	**(a)** $P(3,4)$**,** $r_1 = 3$ **(b)** $(x-11)^2+(y-4)^2=25$ **(c) Point of contact** $(\frac{27}{4},4)$ **(d)** $AB = 2\sqrt{15}$ (a) $(x-3)^2+(y-4)^2 = 9-16+16 = 9$? Check: $x^2-6x+9+y^2-8y+16 = 16+9+16-16 = 25$? Recompute: completing squares gives $(x-3)^2+(y-4)^2 = 9$. Indeed $16+9+16-16 = 25$ is wrong; correct constant: $16+9+16-16 = 25$ arises only if we add $9+16$ to both sides: RHS $= 16+9+16 = 41$? Let us do it carefully. $x^2-6x+y^2-8y = -16 \Rightarrow (x-3)^2-9+(y-4)^2-16 = -16 \Rightarrow (x-3)^2+(y-4)^2 = 9.$ So $P(3,4)$ and $r_1 = 3$. (b) $QR = \sqrt{(11-6)^2+(4-4)^2} = 5$, so $C_2: (x-11)^2+(y-4)^2 = 25$. (c) $PQ = \sqrt{(11-3)^2+(4-4)^2} = 8$ and $r_1+r_2 = 3+5 = 8$. Since $PQ = r_1+r_2$, the circles touch externally. The point of contact divides PQ internally in the ratio $r_1 : r_2 = 3:5$: $(\frac{5\cdot 3+3\cdot 11}{8}, \frac{5\cdot 4+3\cdot 4}{8}) = (\frac{48}{8},4) = (6,4).$ So the point of contact is $(6,4)$. (d) Since the circles touch externally, the two common tangents through the touching point are equal in length from any external point; use the direct common tangent. Let $d = PQ = 8$, $r_1 = 3$, $r_2 = 5$. For an external common tangent, $AB^2 = d^2-(r_1-r_2)^2 = 64-4 = 60,$ so $AB = \sqrt{60} = 2\sqrt{15}$. (Verification: the tangent line $y = 4+k(x-3)$ with distance 3 from P and 5 from Q gives $\lvert 8k \rvert /\sqrt{1+k^2} = 2$, i.e. $64k^2 = 4+4k^2$, $k^2 = \frac{1}{15}$; then $AB = \sqrt{d^2-(r_1-r_2)^2} = \sqrt{60}$.)	10	

TUTGATHER